MATHEMATICS FOR CIRCUITS

W. Chellingsworth

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INTRODUCTORY MONOGRAPHS IN MATHEMATICS

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MATHEMATICS FOR CIRCUITS

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PREFACE

A member of the Royal Society once wrote

> '. . . besides the great delight and pleasure there is in these mathematical and philosophical enquiries there is also much real benefit to be learned, particularly for such gentlemen as employ their estates in those chargeable adventures of draining mines, coal-pits etc. . . .'

This is enough to make any mathematician throw up his hands in horror; but it is a hard fact that the elegant language of mathematics has become the medium through which the scientist and engineer express their problem in a quantitative fashion. Mining engineering may have been uppermost in the mind of our seventeenth century philosopher but the sentiments expressed apply to every branch of applied science.

Electrical engineering is one branch in which the subject matter may be broadly divided under two main headings: the study of field theory, and the study of circuit theory. Each facet requires a sound knowledge of mathematical principles: each requires the same basic approach. First, an appraisal of the nature of the problem, second, a knowledge of the physical laws relating to the particular study in hand, third, the setting out in mathematical form of the relationships which describe the mode of behaviour of the system, and finally, the extraction of an intelligent solution.

It is in respect of this final point that the broad training in engineering is of such vital importance. In the analysis of a circuit problem, for example, it may very well be that two or more solutions will be found in calculating the value of a particular component: it may equally well be that the use of one of the possible values will result in the circuit becoming a charred mass on being put into service. In other words, at least one function of the engineer is to accept the advice offered by the mathematical solution and reject that which is quite irrelevant.

In the following text there is an introduction to the ways in which certain well-known mathematical processes may be applied to circuit theory. An attempt is made to show how simultaneous equations, differential and integral calculus, certain forms of differential equation and complex numbers all find application in a study of networks.

There is an indication that mechanical systems may have the same form of response as certain networks simply because the equations which describe their respective modes of behaviour are identical.

There can be no great depth of study in such a short introductory text; many volumes have already been written and will continue to be written about the response of networks to certain forms of disturbance. This is simply a brief indication of the kind of tools which are necessary to any student of engineering before a deep and detailed study of the subject can be attempted.

I should like to express my gratitude to Mr. A. J. Moakes and Mr. E. H. Williams for the extremely helpful and constructive suggestions which they have made, and to Miss E. Dewdney for preparing the final draft.

1963 W. J. C.

1

SIMULTANEOUS EQUATIONS AND RESISTIVE NETWORKS

1.1 Introduction

The reader probably met simultaneous equations at first in examples of this type:

'If $5x + 3y = 11$ and $9x - 2y = 5$ determine the values of x and y.'

There was no question, at this stage, of having to formulate a problem. The technique of numerical solution was all that was required and the numbers which were obtained had no practical reference. At the next stage, a problem like this may have been tackled:

> 'The weekly wages of four men and three girls working in an office total ninety pounds. When two men leave and three more girls are taken on, the weekly wage bill is unaltered. What is the weekly wage (a) of a man and (b) of a girl?'

Here the reader learned first to translate a given practical situation into algebraic terms and, after solving, to translate the mathematical results into real terms. This is a three-part process which is found to hold through the whole field of applied mathematics.

An important part of this field is the study of electrical circuits. In such a study it may be necessary to set out not just two, but possibly many simultaneous equations which describe the behaviour of a circuit; but before we can do this we must have a knowledge of the basic laws and concepts which apply. These will be found in the following sections.

1.2 Conduction of electricity: potential difference: work and power

Substances which are said to be good conductors of electricity have a molecular structure such that so-called 'outer shell' electrons are relatively freely interchanged between atoms. This interchange is random in nature but by subjecting the material to an electric field a gradual 'drift' of electrons may take place along the conductor. For example, if a battery is connected between two points on the conductor

the electron drift is towards the positive terminal. Since electrons are small negative charges, this implies that there is in effect a transfer of charge in the conductor between the terminals. Associated with this transfer is the production of heat within the conductor; in the example considered a certain amount of energy is extracted from the battery and converted into heat.

In order to carry out a study of cause and effect on a quantitative basis, the transfer of charge and the transfer of energy must in some way be related. A *potential difference* is said to exist between the two points. It is defined as the work done in transferring unit charge from one point to the other. In general terms if an element of charge δq is transferred and δW is the element of energy and v is the potential difference between the two points (assumed constant during the passage of charge), then

$$\delta W = v\,\delta q.$$

The rate at which charge is transferred between the two points is said to be the current in the conductor. In the calculus notation; at any instant of time

$$i = \mathrm{d}q/\mathrm{d}t.$$

The rate of change of energy with respect to time in any system is the power put into, or taken out from, that system. In this case there is continuous power dissipation in the conductor resistance and again, at any instant of time,

$$p = \mathrm{d}W/\mathrm{d}t = v\,\mathrm{d}q/\mathrm{d}t = vi.$$

1.3 Conduction: the idea of resistance and the circuit laws

Ohm's law states that the potential difference between any two points on certain simple types of conductor held at a constant temperature is proportional to the current in the conductor. Or:

$$V \propto I$$

which may be written

$$V = RI$$

where R is the conductor resistance. Now it is a fact that there are conducting materials in which the value of resistance is not constant but dependent upon the magnitude of the current. We must not in general terms, therefore, call the resistance a 'constant' of proportionality. In most cases, however, we will be considering 'linear' resistors and it may be taken, unless otherwise stated, that

$$V = RI.$$

Two other basic laws are necessary, both due to Kirchhoff. The simplest practical expression of these laws is as follows.

(i) The algebraic sum of the currents meeting at a point in a network is zero. This stems from the fact that there can be no accumulation of charge at a point in a network; the charge entering the point must be the charge leaving that point. Since current is the rate of change of charge, the law $\sum I = 0$ expresses this fact.

(ii) The algebraic sum of all changes in potential round any closed path in a network is equal to the algebraic sum of the electromotive forces acting. A practical expression of this law is

$$\sum E = \sum IR.$$

We may regard electromotive force as that force which tends to cause movement of charge in a circuit. The current in a circuit will be taken as flowing in a positive direction from the positive to the negative terminal. It should be noted that this direction, the conventional representation, is opposite to that of electron flow.

Capital letters will be used in applying these laws to resistive circuits, since this is the usual practice when setting up the steady state voltage equations of such circuits. If values at any instant of time were considered the three laws would take the form:

$$v = iR$$
$$\sum i = 0$$

and

$$\sum e = \sum iR.$$

1.4 Power dissipation in resistance

Since the power in a resistive circuit is given by

$$P = VI$$

and since

$$V = RI$$

then

$$P = I^2R$$

or equally

$$P = V^2/R.$$

1.5 Units

It should be understood that in all our equations the symbols refer to measurements made in the appropriate set of units. It is appropriate to detail at this stage the practical units used in such a study and the relationships between them. The primary constants of *mass*, *length* and *time* will be *kilogram*, *metre* and *second* respectively.

From these it is possible to define the unit of *force*. The *Newton* is that force which acting on a mass of one kilogram produces an acceleration of one metre per second per second.

Current is defined with reference to the mutual force which exists between two parallel current carrying conductors. The *Ampere* is the

value of a constant current which when maintained in two parallel rectilinear conductors of infinite length and negligible cross-section, and separated by a distance of 1 metre in vacuo, would produce between these conductors a force equal to 2×10^{-7} newtons per metre.

The *Joule* is the practical unit of energy. It is the energy expended when a force of one newton moves its point of application through a distance of one metre.

The *Coulomb* is the charge transferred in one second when the mean current is one ampere. The practical unit of potential difference, the *Volt*, then follows from the definition given in Section 1.2. From this it can be seen that voltage is a measure of the number of joules of work done per coulomb of charge passed from one point to another.

TABLE 1.1 Summary of practical units

Quantity	*Unit*
Mass	Kilogram
Length	Metre
Time	Second
Force	Newton
Current	Ampere
Energy	Joule (= newton-metre)
Power	Watt (= joule per second)
Charge	Coulomb (= ampere-second)
Potential difference or E.M.F.	Volt (= joule per coulomb)
Resistance	Ohm (= volt per ampere)

The *Ohm* may now be defined as that unit of resistance for which a potential difference of one volt results in a current of one ampere. A summary of these units is given in Table 1.1. Other units will be introduced at appropriate places in the text.

1.6 Circuit reduction

The circuit laws may now be applied to the simple cases of series and parallel connexions of resistors.

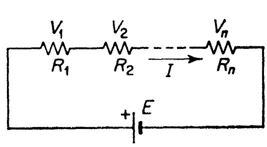


FIGURE 1.1

Series connexion (Figure 1.1). Applying Ohm's Law to each of the series elements

$$V_1 = IR_1$$
$$V_2 = IR_2$$

and in general $$V_n = IR_n.$$

Hence, equating the sum of the potential differences to the source e.m.f.

$$E = I(R_1 + R_2 + \cdots + R_n).$$

Parallel connexion (Figure 1.2). In this case the source voltage is common to all branches

$$I_1 = \frac{E}{R_1}$$

$$I_2 = \frac{E}{R_2}$$

and in general $$I_n = \frac{E}{R_n}$$

Since $$I = I_1 + I_2 + \cdots + I_n$$

then $$I = E\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\right).$$

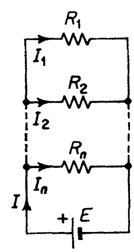


FIGURE 1.2

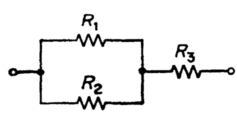


FIGURE 1.3

Series-parallel connexion (Figure 1.3). By using these basic relationships certain circuits may be reduced to a simple series equivalent. For example if two resistors R_1 and R_2 are connected in parallel

$$I = E\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= E\left(\frac{R_1 + R_2}{R_1 R_2}\right)$$

$$\frac{E}{I} = \left(\frac{R_1 R_2}{R_1 + R_2}\right).$$

This is the value of resistance which, connected between the source terminals, would result in the same source current (or power dissipation). If another resistor were placed in series with this arrangement the total effective resistance would be reduced to

$$R = \frac{R_1R_2}{R_1 + R_2} + R_3 \qquad \text{(Figure 1.3)}$$
$$= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1 + R_2}.$$

This so-called 'circuit reduction' is a powerful tool in the solution of network problems. The same example could equally well be solved by direct methods making use of the laws previously set out.

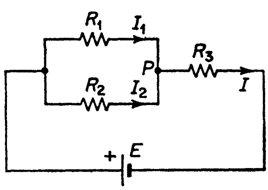


FIGURE 1.4

Kirchhoff's first law applied to the point P (Figure 1.4) gives

$$I_1 + I_2 - I = 0.$$

In establishing this equation, currents entering the point P have been taken as positive; hence the negative sign for I. Then, using the second law and working round the circuit from the source and through R_1 and R_3 in a clockwise direction,

$$E = I_1R_1 + IR_3 \tag{1}$$

and, in the same way but round the circuit which includes R_2 and R_3

$$E = I_2R_2 + IR_3. \tag{2}$$

Since at point P

$$I_2 = (I - I_1)$$

equation (2) reduces to

$$E = -I_1R_2 + I(R_2 + R_3). \tag{3}$$

I_1 may be eliminated between equations (1) and (3) to give

$$\frac{E}{I} = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1 + R_2}$$

which is precisely the same relationship obtained by a process of circuit reduction. This may appear to be somewhat more tedious than the previous method, but it is a fact that very little extra complication is needed to make the apparently straightforward process of reduction somewhat difficult to carry out.

For example, consider the network shown in Figure 1.5. There are methods available for reducing this to an equivalent series resistance.

At this stage, however, we can only set out the circuit equations. In noting the currents in the various branches Kirchhoff's first law has been applied directly without writing down the current equations. The unknown currents have been taken as I_1, I_2 and I_5 with the arbitrary directions shown. In a given case I_5 may not be in that direction, but this is of no concern in the analysis, it simply means that when actual values are substituted, I_5 will come out as a negative quantity. Having specified these currents, the various junctions of the network are taken in turn and the currents in R_3 and R_4 established. The correct evaluation of currents in this or any more complex network can always be checked. The source current at P must be the sum $(I_1 + I_2)$. Similarly the current going back into the source at Q is $(I_1 - I_5) + (I_2 + I_5)$ which is, of course $(I_1 + I_2)$. The voltage equations may now be set out. First, from the source through R_1 and R_4

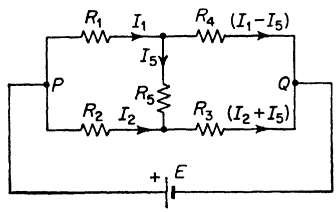


FIGURE 1.5

$$\begin{aligned} E &= I_1R_1 + (I_1 - I_5)R_4 \\ &= I_1(R_1 + R_4) - I_5R_4. \end{aligned} \tag{4}$$

Then, from the source through R_2 and R_3

$$\begin{aligned} E &= I_2R_2 + (I_2 + I_5)R_3 \\ &= I_2(R_2 + R_3) + I_5R_3. \end{aligned} \tag{5}$$

Since there are three unknown currents a third equation is necessary. This might be obtained in several ways. For example:

(i) from the source through R_1, R_5 and R_3 back to the source; or

(ii) from the source through R_2, R_5 and R_4 back to the source.

In fact, neither of these will be used. Instead, take the closed network which includes R_1, R_5 and R_2. In this case, if the voltage drops are taken as positive in a clockwise direction round the 'mesh', and since there is no included e.m.f.

$$I_1R_1 + I_5R_5 - I_2R_2 = 0. \tag{6}$$

Thus, solution of equations (4), (5) and (6) will give the currents in all branches of the network. Had alternative 'routes' been taken through the network to establish the third equation the result would have been the same. All possible voltage equations are implicit in these three. There is little virtue in going through the sheer mechanics of eliminating the variables and solving the above equations; establishing the basic equations is the most important part of

the exercise. However a familiar type of circuit becomes apparent if conditions are such that $I_5 = 0$. In that event the equations can be reduced to

$$I_1(R_1 + R_4) = E = I_2(R_2 + R_3)$$

and

$$I_1R_1 = I_2R_2$$

between these

$$\frac{R_1}{R_1 + R_4} = \frac{R_2}{R_2 + R_3}$$

or

$$R_3R_1 = R_2R_4.$$

The circuit is in fact that of a balanced Wheatstone bridge and these are the balance conditions. Thus a particular case has been picked out of a set of general equations. The example shows that even such a familiar and comparatively simple device as a four-arm bridge presents quite a problem if anything other than the balance condition exists.

It would be possible to choose a variety of circuits in order to illustrate the application of these simple principles. The examples chosen indicate that the solution of such problems involves two main processes.

(i) The setting up of a sequence of equations which includes all the unknowns in the network, and

(ii) the extraction of the unknowns, one by one, by the mechanical process of solving simultaneous equations.

The first process is one which could always be applied by using the basic principles of Ohm and Kirchhoff, but even in this respect many other rules have been devised which facilitate the setting out of circuit equations. To take one example: the networks chosen were represented as having a certain current in each branch.

Alternatively the principle of 'cyclic' currents may be used. Suppose that one imagined an extensive 'grid' of resistors of the form shown in Figure 1.6, with sources interspersed at various points through the grid. If one commenced at the top left-hand corner of this grid and began putting in branch currents the application of Kirchhoff's first law at each junction would result eventually in an array of interdependent currents in each branch. If, however, a cyclic current is represented in each mesh there will be as many currents as there are meshes. In setting out the voltage equation for each circuit the current in any branch which is mutual to two meshes is the difference between the two adjacent cyclic currents.

Thus the current from point A to point B is $(I_1 - I_2)$ and the corresponding potential difference is $(I_1 - I_2)R_{AB}$. The current from point X to point Y is $(I_Q - I_R)$ and the corresponding potential

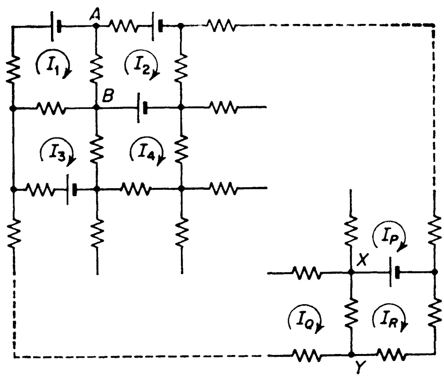


FIGURE 1.6

difference is $(I_Q - I_R)R_{XY}$. The voltage equation for each mesh is set out, as before, by applying Kirchhoff's second law: there will be as many equations as there are cyclic currents. The cyclic current representation for the two circuits previously considered is shown in Figures 1.7 and 1.8 and for other examples in Figures 1.9 and 1.10.

The second process, that of extracting the unknowns from the equations, is dependent upon the tools available to the user. If he has access to a device which solves an array of simultaneous equations, so much the better, although he may have to spend time setting them out in a manner acceptable to the device. Failing this, the process would involve gradual elimination of the unknowns; a purely mechanical process but somewhat tedious if there are more than, say, three equations.

These considerations lead the student of more advanced circuit theory to certain mathematical techniques and concepts which ease the 'machining' of a set of equations. That is why the subject of determinants and, more especially, the subject of matrix notation, figure prominently in any detailed work on circuit analysis (see Appendix II).

One important point should be made in connexion with the notes set out so far: in all cases the principle of superposition has been assumed to apply. The resistive elements have been assumed to be 'linear'. Suppose that a source of e.m.f. E_1 is applied to a network and that the resultant current from the source is I_1. Now suppose that this source is removed and a second source E_2 is applied resulting in a current into the network of I_2. If the application of a source e.m.f. of $(E_1 + E_2)$ results in a current from the source of $(I_1 + I_2)$ then the network is linear and the principle of superposition holds.

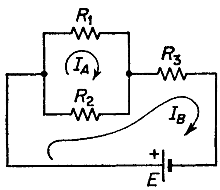


FIGURE 1.7

Equations

$$I_A R_1 + (I_A - I_B)R_2 = 0$$
$$(I_B - I_A)R_2 + I_B R_3 = E.$$

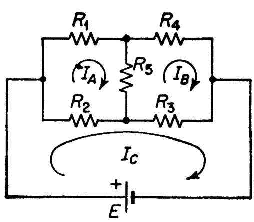


FIGURE 1.8

Equations

$$\begin{aligned} I_A R_1 + (I_A - I_B)R_5 & \\ + (I_A - I_C)R_2 &= 0 \\ I_B R_4 + (I_B - I_C)R_3 & \\ + (I_B - I_A)R_5 &= 0 \\ (I_C - I_A)R_2 & \\ + (I_C - I_B)R_3 &= E. \end{aligned}$$

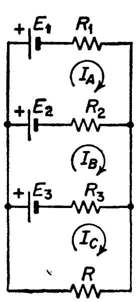


FIGURE 1.9

Equations

$$\begin{aligned} I_A R_1 + (I_A - I_B)R_2 & \\ &= E_2 - E_1 \\ (I_B - I_A)R_2 + (I_B - I_C)R_3 & \\ &= E_3 - E_2 \\ (I_C - I_B)R_3 + I_C R &= -E_3. \end{aligned}$$

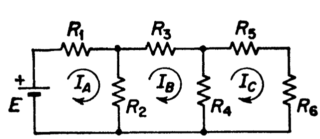


FIGURE 1.10

Equations

$$\begin{aligned} I_A R_1 + (I_A - I_B)R_2 &= E \\ (I_B - I_A)R_2 + I_B R_3 & \\ + (I_B - I_C)R_4 &= 0 \\ (I_C - I_B)R_4 & \\ + I_C(R_5 + R_6) &= 0. \end{aligned}$$

1.7 Non-linear resistor

It may be that the voltage across a resistor is not directly proportional to the current through it but follows some law expressed in a general way as

$$V = F(I).$$

That is, the voltage is given by some function of the current. A commonplace example of this is the domestic filament lamp. Such elements are termed non-linear resistors, and it should be apparent from the following example that analysis in general terms is made more difficult by this variation of resistance with current.

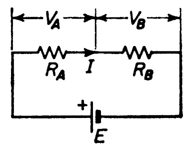


FIGURE 1.11

Suppose that a battery of e.m.f. E is applied to two resistors R_A and R_B in series (Figure 1.11), we would at once say that the current supplied from the battery is

$$I = \frac{E}{R_A + R_B}.$$

A more formal way of stating the problem is

$$\begin{aligned} E &= V_A + V_B \\ V_A &= IR_A \\ V_B &= IR_B \end{aligned}$$

which amounts to the same thing. The formal way would have to be adopted when dealing with a non-linear element. In this event the equations would take the form

$$E = V_A + V_B \tag{7}$$
$$V_A = IR_A$$
$$V_B = F(I). \tag{8}$$

The non-linear element is the resistor B. Equation (7) may be written as

$$\begin{aligned} V_B &= E - V_A \\ &= E - IR_A. \end{aligned} \tag{9}$$

Expressed in this way (8) and (9) are both relationships between V_B and I, that is, we have arrived at a pair of simultaneous equations. One is the equation of a straight line, the other may not readily be expressed as a mathematical function but may be presented as a graphical relationship between voltage and current. There will be one, or possibly more than one, value of V_B and I which satisfy both relationships. By plotting V_B for each equation for a chosen range of I, the intersection of the two characteristics determines the operating

point. All relevant information (V_B, I and V_A) may now be extracted and a complete solution obtained (Figure 1.12).

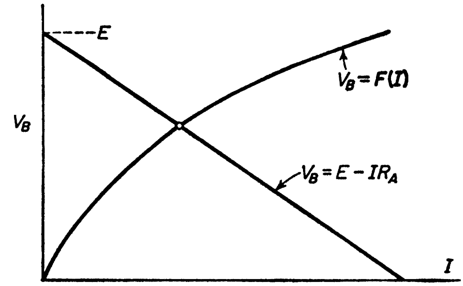


FIGURE 1.12

It may be that the resistor R_A is varied over a certain range. In order to establish the various values of current and the relevant points on the characteristic, consequent upon this, all that is required is a 'family' of straight lines corresponding to the various values of R_A. These would be known as 'load lines' (Figure 1.13).

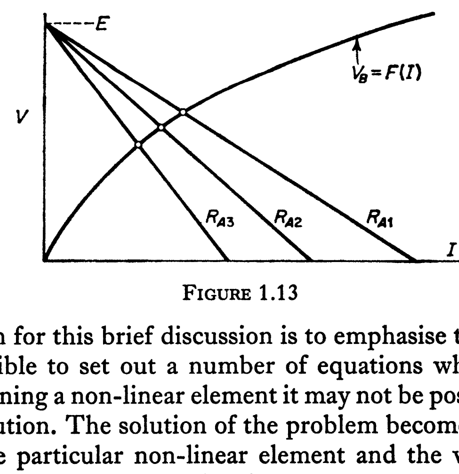


FIGURE 1.13

The reason for this brief discussion is to emphasise that although it may be possible to set out a number of equations which apply to a circuit containing a non-linear element it may not be possible to extract a general solution. The solution of the problem becomes one which is special to the particular non-linear element and the values of linear elements with which it is associated.

Exercises

1 See Figure 1.4

If

$$R_1 = 10 \text{ ohms}$$
$$R_2 = 8 \text{ ohms}$$
$$R_3 = 5 \text{ ohms}$$

and a current of 10 amperes flows in R_1 determine:

(a) the potential difference across the parallel branches
(b) the current I_2
(c) the current I
and (d) the value of E.

Reduce the circuit to an equivalent series resistance R and check that $E = IR$.

2 See Figure 1.5

If

$$R_1 = 10 \text{ ohms}$$
$$R_4 = 100 \text{ ohms}$$
$$R_3 = 5 \text{ ohms}$$

Determine

(a) the value of R_2 which will give balanced conditions. If R_2 is now altered to 1·0 ohm it is found that

$$I_1 = 10 \times 10^{-3} \text{ amperes}$$

and $I_2 = 160 \times 10^{-3}$ amperes, determine

(b) the current in R_5
(c) the potential difference across R_5
(d) the value of E.

3 See Figure 1.10

If all values of resistance are 1·0 ohm and $E = 10$ volts

(a) reduce this circuit to a single series resistor, and
(b) evaluate the total current input from the source.

Formulate the circuit equations and check the above answers by solving the equations for I_A.

4 See Figure 1.9

In this circuit

$$E_1 = E_2 = 2 \text{ volts}$$
$$E_3 = 4 \text{ volts}$$
$$R_1 = R_2 = R_3 = 1{\cdot}0 \text{ ohm}$$
$$R = 5 \text{ ohms.}$$

Evaluate

(a) the load current
(b) the current supplied from each source
(c) the power dissipated in R.

5 See Figure 1.11

A non-linear resistor R_B is defined by

$$V = 30 \times 10^3 \times I^2$$

where V is in volts and I in amperes. Plot this characteristic for a range of current up to $I = 20 \times 10^{-3}$ amperes and, taking E as 12 volts determine the circuit current for the following values of R_A

(a) 600 ohms
(b) 800 ohms
(c) 1200 ohms.

6 The non-linear resistor defined in question (5) is connected in parallel with a constant resistance of 500 ohms. This parallel combination is connected in series with a constant resistance of 800 ohms. By considering values of current in the non-linear resistor up to $I = 20 \times 10^{-3}$ amperes, plot a characteristic showing the relationship between the total applied voltage and the total input current to the circuit.

2

APPLICATIONS OF DIFFERENTIAL AND INTEGRAL CALCULUS

2.1 Determination of a maximum value

Ask a student of calculus in the early stages of his study how this branch of mathematics may be used, and it is likely that he will think first of the finding of maximum or minimum values of variable quantities. Our previous work will immediately afford a simple example of this.

Suppose that a source of e.m.f. E having an internal resistance R_1 is connected to an external load resistor R_2 which can be varied in magnitude. Then the current flowing in the circuit is given by

$$I = \frac{E}{R_1 + R_2}$$

and the power dissipated in the load resistor is

$$\begin{aligned} P &= I^2 R_2 \\ &= \frac{E^2 R_2}{(R_1 + R_2)^2}. \end{aligned}$$

If the value of R_2 is put equal to zero the power dissipated will be zero; if it is given an infinite value, the power will again be zero.

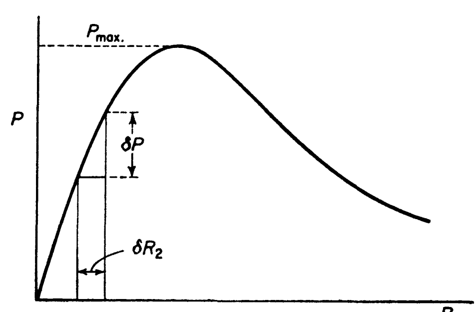


FIGURE 2.1

Somewhere between these two extremes the power dissipated will have a maximum value. For given values of E and R_1 a graph may be plotted relating the power dissipated to the external resistance value. If this is changed by an amount δR_2 the power will change by an amount δP. The small change in power with respect to the corresponding change in resistance is given by $\delta P/\delta R_2$ (Figure 2.1).

As the increment considered becomes vanishingly small this value tends to the gradient of the tangent to the curve at the point considered so that $\mathrm{d}P/\mathrm{d}R_2$, in the limit, gives the slope of the characteristic. The maximum value of P corresponds to zero slope. Thus in order to determine the condition for which the power dissipated is a maximum, $\mathrm{d}P/\mathrm{d}R_2$ must be evaluated and equated to zero.

$$P = \frac{E^2 R_2}{(R_1 + R_2)^2}$$

therefore

$$\frac{\mathrm{d}P}{\mathrm{d}R_2} = \frac{E^2 \times 1 \times (R_1 + R_2)^2 - 2E^2R_2(R_1 + R_2)}{(R_1 + R_2)^4}.$$

For this expression to be zero we must have $R_2 = R_1$. The form of the curve indicates that P is a maximum (and not a minimum) and by substitution we have

$$P_{\text{max}} = \frac{E^2}{4R_2}.$$

Thus the maximum power will be dissipated in the load when the load resistance is equal to the source resistance. This is known as the *maximum power transfer theorem.*

There are many such instances where differential calculus enables us to determine the optimum value of a circuit parameter. However, probably the most useful application of calculus to circuits is related to cases where quantities are varying with respect to time; where current, or voltage or charge has a certain rate of change at an instant of time.

2.2 Time variations

When a quantity x is changing with respect to time t we say that the rate of increase of x at any instant is $\mathrm{d}x/\mathrm{d}t$. If x is the displacement of a body from some datum then $\mathrm{d}x/\mathrm{d}t$ is the velocity of the body; such notation is commonplace when dealing with problems in dynamics. Equally, however, there may be rates of change in an electric circuit. A battery, connected to a circuit through a switch, supplies no current when the switch is open but a current is established when the switch is closed. At one instant there is zero current, the next, a finite value

of current. There has been, for a time, a certain rate of increase of current.

But what determines how fast the current rises to a new value? Has it changed at an infinite rate in zero time or has it reached a finite value in a matter of microseconds? Up to this point only resistive circuits have been considered and the only parameter which has concerned us has been resistance; but circuits may have other properties which assume importance when circuit quantities are changing. Inductance and capacitance are two such properties. A brief introduction to these properties now follows.

2.3 Inductance

When a current passes in a loop of conductor a magnetic flux is established which threads the loop, lines of flux forming closed paths which embrace the conductors forming the loop. The magnitude of the flux is determined by the magnitude of the current and if the medium in which the flux is established is non-magnetic, current and flux are directly proportional. The loop may consist of a number of turns and in these circumstances we may say that the flux linkages ΦN are proportional to i, where Φ is the flux linking with N turns of a loop carrying current i. This may be written as

$$\Phi N \propto i$$

or

$$\Phi N = Li$$

where L, the constant of proportionality, is known as the inductance of the system. One must be careful about talking of 'constants' of porportionality in any physical system; that is why reference was made to the non-magnetic nature of the medium. If the medium is magnetic; if for example the turns of the loop are wound on an iron core, this direct proportionality between current and flux may no longer exist. Although it is true to say that the system is inductive, it will, of necessity, have a value of inductance at a particular value of current which may not be the same at a different value of current. The reason for this departure from proportionality is bound up with the magnetic properties of ferrous materials. So-called 'saturation' of the iron core may occur at higher values of current resulting in a condition whereby further increase in current will not appreciably increase the flux. The inductance of any arrangement of conductors is a function of the geometry of the system; the number of turns and the area enclosed by the loop. An overhead line along which electric power is transmitted possesses inductance (it is, effectively, an elongated loop) so also does a coil of wire or the winding of an electric motor. Our function is not, however, to evaluate from the physical

properties of the system the inductance of the conductor arrangement, although this in itself makes an interesting mathematical exercise. It is rather to examine the effect of inductance in circuits.

If the flux linking with a conductor system is changing, an e.m.f. is induced which is directly proportional to the rate of change of flux linkage. Considering a coil of N turns, for example, the induced e.m.f. is given by

$$e = -\frac{\mathrm{d}(\Phi N)}{\mathrm{d}t}.$$

The negative sign indicates that the e.m.f. is in such a sense as to oppose the change in flux.

Since $$\Phi N = Li$$

the e.m.f. may be expressed as

$$e = -L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

Thus the rate of change of current in an inductive system has an induced e.m.f. associated with it. This is an important effect in any consideration involving electrical circuits. It means that if a constant steady value of current flows in an inductive circuit we are not at all concerned with induced e.m.f.'s; but if the current is changing, we are.

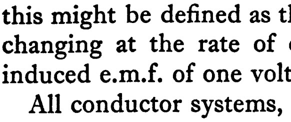


FIGURE 2.2

The practical unit of inductance is the *Henry*. From the e.m.f. equation this might be defined as the inductance of a circuit in which current, changing at the rate of one ampere per second, results in a self-induced e.m.f. of one volt.

All conductor systems, unless they are arranged in a certain way, inherently possess a certain amount of inductance. Suppose that a voltage is applied to a circuit having resistance R and inductance L (Figure 2.2). When the switch is open the current in the system is considered to be zero. When the switch is closed the current increases at a certain rate. We may say that the source voltage is E but that there is now an additional 'seat' of e.m.f. e. Thus the total e.m.f. $(E + e)$ serves to establish the current in circuit resistance. Applying Kirchhoff's second law,

$$E + e = iR$$

hence $$E - L\frac{\mathrm{d}i}{\mathrm{d}t} = iR$$

or
$$E = L\frac{\mathrm{d}i}{\mathrm{d}t} + iR.$$

This is a first order differential equation describing precisely the manner in which the current changes with time. But one should never solve such an equation without extracting from it, in its present form, the maximum amount of information. For example we have specified that at $t = 0$, $i = 0$. It follows that at $t = 0$

$$E = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

or
$$\frac{\mathrm{d}i}{\mathrm{d}t}_{t=0} = \frac{E}{L}.$$

Thus we know the initial rate of change of current on closing the switch. Now suppose that we specify that all rates of change have vanished which is tantamount to saying that steady state conditions exist. In this event $\mathrm{d}i/\mathrm{d}t = 0$ and

$$E = iR$$

or
$$i = E/R.$$

Thus without attempting to solve the differential equation we have established that:

(i) The current will have a certain initial rate of change.

(ii) The current will have a certain steady value.

The significance of this we shall examine later. For the moment having seen that circuits may possess inductance as well as resistance, let us examine a third property of circuits, capacitance.

2.4 Capacitance

In its simplest form a capacitor may be considered as a pair of conductors in air, or separated by an insulating medium. If a potential difference exists between the conductors an electrostatic field is established; the system is said to be charged. Associated with the establishment of a charge is an electric current, the relationship between charge q and current i at any instant being that $i = \mathrm{d}q/\mathrm{d}t$.

Furthermore, the magnitude of the charge is related to the potential difference v between the conductors. In the majority of systems this may be taken to be a proportional relationship expressed as $q \propto v$, or $q = Cv$ where C is known as the capacitance of the arrangement.

The practical unit of capacitance is the *Farad*. This might be defined as the capacitance of a system in which a potential difference of one volt results in a transfer of charge of one coulomb. As in the

case of inductance the value of C is determined by the geometry of the arrangement and by the insulating medium in which the flux, in this case the electrostatic flux, is established. Again the evaluation of capacitance of a given arrangement makes an interesting mathematical exercise but, again, we are more concerned with the effect of capacitance in circuits than with the value of any physical system.

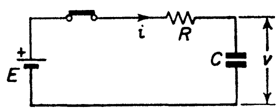


FIGURE 2.3

The term capacitor may be associated in the mind with the type of component met with in radio construction. But it should be remembered that any system comprising an arrangement of conductors separated by an insulating medium possesses this property. Overhead power lines have capacitance and there is inter-turn capacitance between adjacent turns of an inductive winding.

Now suppose that such a capacitor, whatever form it takes, is connected to a source of e.m.f. through a resistor (Figure 2.3) and that before the switch is closed the capacitor is uncharged. If q is the instantaneous charge on the capacitor this condition means that at $t = 0$, $q = 0$.

Then, when the switch is closed, by equating the source e.m.f. to the potential differences summed round the circuit.

$$E = iR + v.$$

But

$$q = Cv$$

and

$$i = \frac{dq}{dt}$$

hence

$$E = R\frac{dq}{dt} + \frac{q}{C}.$$

As before, as much information as possible should be extracted from this equation before any attempt is made to find a general relationship between charge and time. At $t = 0$, $q = 0$, hence

$$\frac{dq}{dt}_{t=0} = \frac{E}{R}.$$

This gives the initial value of current on closing the switch.

Furthermore, when all rates of change have vanished, that is when $dq/dt = 0$

$$E = q/C$$

or

$$q = CE$$

which gives the steady state charge on the capacitor.

2.5 Comparison of equations

Two distinct circuits, one inductive the other capacitive have been examined. Suppose that the two equations are set out together.

$$E = L\frac{\mathrm{d}i}{\mathrm{d}t} + iR \quad \text{or} \quad \frac{E}{R} = \frac{L}{R}\frac{\mathrm{d}i}{\mathrm{d}t} + i \tag{1}$$

and

$$E = R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} \quad \text{or} \quad CE = CR\frac{\mathrm{d}q}{\mathrm{d}t} + q. \tag{2}$$

These equations (1) and (2) are of precisely the same mathematical form. Physically the systems are quite different but mathematically they may be expressed in a quantitative fashion as satisfying the more general equation

$$X_{\text{s.s.}} = \tau\frac{\mathrm{d}x}{\mathrm{d}t} + x$$

where $X_{\text{s.s.}}$ ($= E/R$ or CE) is the steady state value, τ is a constant and x and t are variables.

The solution of this common equation is an exercise in integration. We may write it as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\tau}(X_{\text{s.s.}} - x)$$

or

$$\frac{\mathrm{d}x}{X_{\text{s.s.}} - x} = \frac{\mathrm{d}t}{\tau}.$$

Integrating both sides

$$\int\frac{\mathrm{d}x}{X_{\text{s.s.}} - x} = \int\frac{\mathrm{d}t}{\tau} + \text{constant}$$

$$-\log_e(X_{\text{s.s.}} - x) + \log_e A = t/\tau.$$

Note that $\log_e A$ has been chosen as the constant term arising out of the integration rather than just A. This is purely for convenience. It follows that

$$\log_e\frac{A}{X_{\text{s.s.}} - x} = \frac{t}{\tau}$$

or

$$\frac{A}{X_{\text{s.s.}} - x} = e^{t/\tau}. \tag{3}$$

The value of the constant A may now be determined from a knowledge of a particular value of x at a particular instant of time. Suppose

that at $t = 0$, $x = 0$. (This agrees with at $t = 0$, $i = 0$ or $q = 0$ in the circuits considered.) Then from (3)

$$\frac{A}{X_{\text{s.s.}}} = 1$$

or
$$A = X_{\text{s.s.}}.$$

Substituting for A in (3) results in

$$\frac{X_{\text{s.s.}}}{X_{\text{s.s.}} - x} = e^{t/\tau}$$

or
$$x = X_{\text{s.s.}}(1 - e^{-t/\tau}).$$

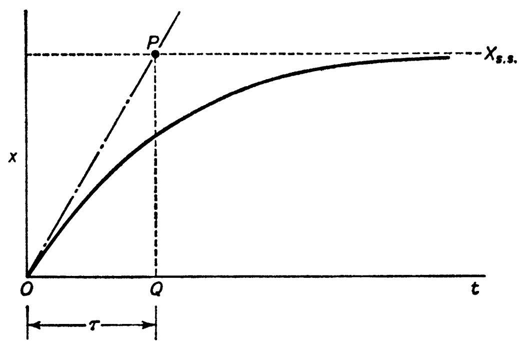


FIGURE 2.4

This shows how x varies with time. Plotting the variation with time gives the response shown in Figure 2.4. It should be noted that if both sides of this final equation are differentiated with respect to time

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{X_{\text{s.s.}}}{\tau} e^{-t/\tau}$$

and at $t = 0$
$$\frac{\mathrm{d}x}{\mathrm{d}t_{t=0}} = \frac{X_{\text{s.s.}}}{\tau}.$$

If this initial rate of change were maintained, as indicated by the line OP in the sketch, the steady value $X_{\text{s.s.}}$ would be reached after time τ, given by OQ on the diagram. τ is said to be the 'time constant' of the system defined by the original differential equation.

Writing down the solutions applying to the two circuits

$$i = E/R \times (1 - e^{-Rt/L})$$
$$q = CE \times (1 - e^{-t/CR}).*$$

* If v is the value of capacitor voltage at any instant, then since $v = q/C$ it follows that $v = E\,(1 - e^{-t/RC})$. The capacitor voltage, therefore, also rises exponentially with time.

Current in the one case and charge in the other rise to a steady value exponentially with time. One other interesting feature of this brief study is that $\tau = L/R$ in the inductive circuit and $\tau = CR$ in the capacitive circuit.

τ, as we have seen, is measured in terms of time and is usually expressed in seconds. Thus inductance and resistance, and capacitance and resistance are related to the dimensions of time.

This reference to inductive and capacitive circuits by no means exhausts the subject—in fact, it barely scratches the surface. It does, however, bring out one important point, namely that mathematics is the common thread which runs through any study of physical systems and that before one examines such systems in a piecemeal fashion it is as well to examine the basic equations which describe them.

Exercises

1 See Figure 2.2

If
$$E = 10 \text{ volts}$$
$$L = 1 \text{ henry}$$
$$R = 200 \text{ ohms}$$

(a) write down the expression for current at any instant of time t following closure of the switch
(b) derive expressions for the instantaneous resistance voltage and the self induced e.m.f.
(c) determine the value of current at the instant 0·005 seconds after switching
(d) determine the rate of change of current at the instant of switching and at the instant 0·005 seconds after switching
(e) by integrating the power loss, evaluate the energy dissipated in the resistor during the first 0·005 seconds.

2 See Figure 2.3

If
$$E = 10 \text{ volts}$$
$$C = 0{\cdot}1 \times 10^{-6} \text{ farads}$$
$$R = 2 \times 10^{6} \text{ ohms}$$

(a) write down the expression for charge at any instant following closure of the switch
(b) determine the magnitude of the charge at the instant 0·2 seconds after switching
(c) derive an expression for instantaneous current

(d) evaluate the current at the instant of switching and at the instant 0·2 seconds after switching
(e) derive an expression for the instantaneous resistor voltage
(f) derive an expression for the instantaneous capacitor voltage
(g) evaluate the energy dissipated in the resistor during the first 0·2 seconds.

3 See Figure 2.3 and values as in question 2

This circuit is again switched on, but this time with the capacitor already charged to a potential difference of 8 volts. Derive expressions for

(a) the instantaneous charge
(b) the instantaneous current

following closure of the switch.

3

SIMPLE HARMONIC MOTION AND OSCILLATORY SYSTEMS

3.1 Torsional pendulum

One of the difficulties about the analysis of circuits is that it is not at all easy to visualise what is going on in the system under consideration. This is not quite so much the case with mechanical devices, where the 'if you push here, this much will happen there' approach helps one to form a mental picture of the problem.

Often it is possible to call upon the tangible mechanical problem to assist in the understanding of the more intangible circuit problem. The study of sinusoidal oscillations is a case in point; simple harmonic motion is something familiar to all of us (from the very first rocking of the cradle). Imagine a mechanical system comprising a heavy flat disc suspended at its centre by means of a thin pliant rod or fibre which is rigidly clamped at the top end (Figure 3.1). If the disc is twisted through a certain angle, energy will be stored in the rod; it possesses so much pent-up, or potential, energy. On release the disc will rotate back towards its original position, gathering speed as the potential energy in the rod is transformed to kinetic energy. When it reaches the original position of rest the disc will possess so much momentum that it will carry on turning, twisting the rod in the opposite direction. The kinetic energy is now converted back to potential energy and so the oscillation continues, in an ideal system the translation of energy back and forth resulting in continuous oscillation at a certain frequency. The value of frequency is related to the physical constants of the system: the compliance of the rod and the inertia of the disc. Setting these considerations out in quantitative form, suppose that:

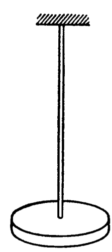


FIGURE 3.1

J = inertia of disc

λ = compliance of the rod: that is, the angle through which the rod twists when unit torque is applied

θ_0 = the angle through which the rod is first twisted from the equilibrium position

θ = the angle at any instant during the oscillation.

Both angles are measured from the equilibrium position of the rod and disc.

$\frac{d\theta}{dt}$ = the rate of change of angular position: that is, the angular velocity

$\frac{d^2\theta}{dt^2}$ = the rate of change of angular velocity: that is, the angular acceleration.

Newton's second law enables us to relate the angular acceleration of a body to the torque which causes the acceleration. In effect it says that the net accelerating torque and the angular acceleration are proportional, the constant of proportionality being the system inertia. In symbols

$$T_a = J\frac{d^2\theta}{dt^2}.$$

Now the net accelerating torque on the system is given by the value of any external torque which may be applied, diminished by any reaction torque which may be acting. In this case there is no external torque applied and the reaction torque is that due to the twisting of the rod. It follows that

$$T_a = 0 - \frac{\theta}{\lambda}.$$

Note that from the definition of compliance an angle of twist θ is associated with a restoring, or reaction, torque of θ/λ. The torque equation now becomes

$$0 - \frac{\theta}{\lambda} = J\frac{d^2\theta}{dt^2}$$

or

$$J\frac{d^2\theta}{dt^2} + \frac{\theta}{\lambda} = 0. \tag{1}$$

A function of θ must be found which satisfies this. There are several pointers as to the nature of the function. For one thing, it has been specified that at the start of it all when $t = 0$, $\theta = \theta_0$. For another, it must be a function which, when differentiated twice, has the same form as the original function. The reason for this will become apparent later.

Suppose that we try the function $\theta = \theta_0 \cos \omega t$, where ω is some, as yet unspecified, constant. Then, at $t = 0$, $\theta = \theta_0$; so that at least the first condition is fulfilled. Now, differentiating twice

$$\frac{d\theta}{dt} = -\omega\theta_0 \sin \omega t$$

and

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta_0 \cos \omega t.$$

Substituting in the original equation (1)

$$J(-\omega^2\theta_0 \cos \omega t) + \frac{1}{\lambda}.\theta_0 \cos \omega t = 0$$

$\theta_0 \cos \omega t$ cancels. (This was the reason why it was essential to have a function which gave a similar form after two successive differentiations.) Furthermore the left hand side of the equation only equals zero provided that

$$J\omega^2 = \frac{1}{\lambda}$$

or

$$\omega = \left(\frac{1}{J\lambda}\right)^{1/2}.$$

The time T of one complete oscillation, given that when $t = T$, $\omega T = 2\pi$, can be expressed as

$$T = \frac{2\pi}{\omega}. \qquad \text{(Figure 3.2)}$$

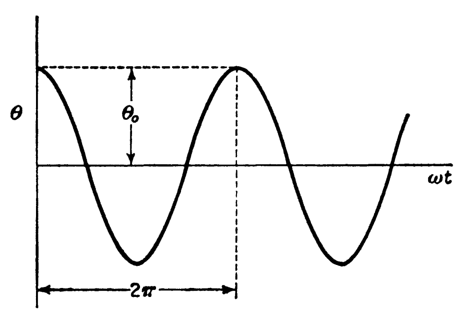


FIGURE 3.2

This is the time of one cycle (usually expressed in seconds) so that the

number of cycles per second, or the frequency of oscillation is given by

$$f = \frac{1}{T}$$

$$= \frac{\omega}{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{1}{J\lambda}\right)^{1/2}.$$

This method of solving the equation of motion is not particularly satisfying from the mathematical viewpoint because recourse to a 'try it and see' method involving an intelligent guess at the solution has to be made. Perhaps a mental picture of the way in which the system oscillates helps in the choice of function; this very often happens in the solution of engineering problems. We have an idea what the answer should be from a knowledge of the nature of the problem; we try a solution and gradually shape it until it satisfies completely the mathematical requirement.

3.2 The oscillatory electrical system

Some little time has been spent on the torsional pendulum but it is a system which is easily visualised and, by analogy, the response of a corresponding electrical circuit may be deduced. Suppose that a capacitor, initially charged, is switched into a purely inductive circuit (Figure 3.3). Let the initial charge on the capacitor be Q and at any instant t after closing the switch suppose that

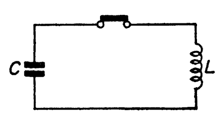


FIGURE 3.3

$$q = \text{capacitor charge}$$
$$\mathrm{d}q/\mathrm{d}t = \text{rate of change of charge} = \text{current } i, \text{ and}$$
$$\mathrm{d}^2q/\mathrm{d}t^2 = \mathrm{d}i/\mathrm{d}t = \text{rate of change of current.}$$

Kirchhoff's second law says, in effect, that the total e.m.f. acting round the circuit is equal to the sum of the potential differences round the circuit. There is, by assumption, no resistance in the circuit and consequently there can be no potential drop due to current in resistance. Between the plates of the capacitor, however, there is a potential difference given by

$$q = Cv$$

or

$$v = \frac{q}{C}.$$

Furthermore, the total e.m.f. round the circuit is made up of any external e.m.f. which may be applied, together with any internal induced e.m.f. From the circuit it can be seen that there is no external source e.m.f. and the induced e.m.f. is given by

$$e = -L\frac{\mathrm{d}i}{\mathrm{d}t}$$

hence the total e.m.f. is

$$0 - L\frac{\mathrm{d}i}{\mathrm{d}t}$$

and equating this to the potential difference

$$0 - L\frac{\mathrm{d}i}{\mathrm{d}t} = v = \frac{q}{C}$$

or

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{q}{C} = 0. \qquad (2)$$

Compare this with equation (1): it has exactly the same form and there is correspondence between the terms J and L, λ and C, θ and q. The initial condition of twist imposed, θ_0, corresponds to the initial capacitor charge Q. By analogy it follows that the charge at any instant is given by

$$q = Q\cos\left(\frac{1}{LC}\right)^{1/2} t.$$

The frequency of oscillation of the charge is given by

$$f = \frac{1}{2\pi}\left(\frac{1}{LC}\right)^{1/2}$$

and since charge and capacitor potential are related by the expression $q = Cv$ the manner in which the voltage varies with time follows directly as

$$\begin{aligned} v &= \frac{q}{C} \\ &= \frac{Q}{C}\cos\left(\frac{1}{LC}\right)^{1/2} t. \end{aligned}$$

Moreover, the current in the circuit could be assessed simply from the fact that $i = \mathrm{d}q/\mathrm{d}t$. However, it is not the object of this exercise to examine in any detail the behaviour of a circuit; it is to emphasise that certain similarities exist in what are, physically, quite different systems. Mathematically, the two systems have identical equations of motion.

3.3 Energy considerations

It has been noted that there are two forms of energy storage in the mechanical system: the potential energy stored in the pliant rod and the kinetic energy possessed by the rotating disc. In the electrical circuit there are still two such forms: that associated with the charged capacitor and that with the inductor. It is a comparatively simple exercise in applied mechanics to show that:

$$\text{The energy stored in twisting a pliant rod} = \frac{1}{2}\frac{\theta^2}{\lambda}$$

$$\text{The kinetic energy of the disc} = \frac{1}{2}J\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$$

If the analogy really holds true it follows that:

$$\text{The energy stored in the charged capacitor} = \frac{1}{2}\frac{q^2}{C}$$

$$\text{The energy stored in the inductor} = \frac{1}{2}L\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 = \frac{1}{2}Li^2$$

These expressions indicate the energy required to establish an electric field between the conductors of a capacitor and to set up the magnetic field consequent upon the passage of current in the inductive winding. The oscillations which we have noted in the electrical circuit are the result of transference of energy from one form to the other.

In both the mechanical and electrical systems the continuous interchange of energy between the two basic components was maintained because nowhere was there any dissipation of energy. But the laws of nature are such that no system can so maintain itself without some form of external impetus; the concept of perpetual motion is not physically realisable. The mechanical system, even if suspended in vacuo, would sooner or later come to rest. What it amounts to is that the energy initially given to the system will have been dissipated in the form of heat in the rod. Some resistance is inevitable in the electrical circuit, and in this case the energy initially stored in the charged capacitor will eventually be dissipated in circuit resistance.

3.4 The effect of damping

In each of the two equations which describe the systems a new term (a so-called 'damping' term) will be introduced. Suppose that a vane,

free to move in a viscous fluid, is attached to the underside of the disc and further suppose that the torque necessary to turn the disc under these circumstances is proportional to the speed of rotation. When the disc is moving this implies that there will be an additional reaction torque given by

$$T_f \propto \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

or

$$T_f = K\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

The torque equation then becomes

$$0 - \frac{\theta}{\lambda} - K\frac{\mathrm{d}\theta}{\mathrm{d}t} = J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

or

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + K\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\theta}{\lambda} = 0.$$

In the circuit the presence of resistance gives rise to an additional drop in potential due to current in the resistance of the circuit; its magnitude is

$$v_R = iR = R\frac{\mathrm{d}q}{\mathrm{d}t}$$

and the voltage equation now becomes

$$0 - L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{q}{C} + R\frac{\mathrm{d}q}{\mathrm{d}t}$$

or

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0.$$

Once again, the analogy between the two systems is apparent and it is possible to set out the relationships as in Table 3.1.

TABLE 3.1

Mechanical		*Electrical*	
Inertia	J	Inductance	L
Friction constant	K	Resistance	R
Compliance	λ	Capacitance	C
Displacement	θ	Charge	q
Velocity	$\mathrm{d}\theta/\mathrm{d}t$	Current	$i(=\mathrm{d}q/\mathrm{d}t)$
Acceleration	$\mathrm{d}^2\theta/\mathrm{d}t^2$	Rate of increase of current	$\mathrm{d}i/\mathrm{d}t$

An indication of the method of solution of this form of equation is set out in Appendix I. At the present stage these equations will be dealt with in a qualitative way. The second term in each, the damping term, will obviously preclude the possibility of the same continuous oscillatory response which applied to the ideal systems previously considered. Consideration of the more tangible mechanical system, with its vane moving in a fluid, would point to the fact that eventually the system must come to rest. In the electrical circuit the energy initially stored in the capacitor must be dissipated in resistance. What is not at all obvious is the manner in which this state is reached.

Both equations may be written in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_n \frac{\mathrm{d}x}{\mathrm{d}t} + \omega_n^2 x = 0$$

where the parameters ω_n and ζ are related to the actual system constants in the way shown in Table 3.2

TABLE 3.2

Parameter	*Mechanical*	*Electrical*
ω_n	$\left(\frac{1}{J\lambda}\right)^{1/2}$	$\left(\frac{1}{LC}\right)^{1/2}$
ζ	$\frac{K}{2}\left(\frac{\lambda}{J}\right)^{1/2}$	$\frac{R}{2}\left(\frac{C}{L}\right)^{1/2}$

The significance of ω_n should be immediately apparent. It relates to the frequency of oscillation of the system in the absence of the damping term. The significance of ζ is not so apparent but a complete solution of the above equation, assuming that at $t = 0$, $x = X_0$, would show a number of modes of response (Figure 3.4). In this case where ζ is greater than unity the response is said to be that of an 'overdamped' system. In other words there will be no tendency to oscillation at all. If, for example, the vane were moving in a thick, treacly fluid one could imagine the mechanical system as having this kind of sluggish approach to a condition of rest. (But bear in mind that the damping constant is not the only factor which determines the value of ζ.) On the other hand if ζ is less than unity the system performs a series of oscillations of constant frequency but diminishing amplitude. It is said to be 'underdamped'. For the particular case of

$\zeta = 0$ continuous oscillation results. The intermediate special case of $\zeta = 1$ is said to give a 'critically damped' response.

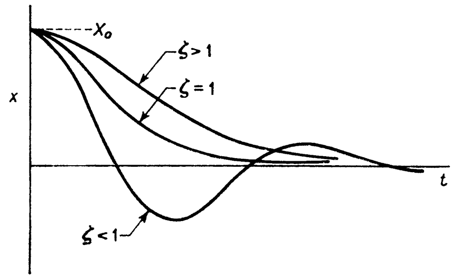


FIGURE 3.4

The whole point of this brief excursion into the realm of differential equations is this: the subject of oscillations is one which forms part of any engineering study. The response of a circuit or a galvanometer, the vibration of a piece of rotating machinery or a strut subjected to a disturbance may be quite different physical phenomena but the actual mode of behaviour may be related to exactly the same form of differential equation. One cannot begin to analyse such systems unless one is adequately equipped in this respect.

Exercises

1 See Figure 3.3

If
$$L = 10 \times 10^{-3} \text{ henrys}$$
$$C = 4 \times 10^{-6} \text{ farads}$$
$$Q = 20 \times 10^{-6} \text{ coulombs}$$

derive expressions for

(a) capacitor charge
(b) capacitor voltage
(c) circuit current
(d) self induced e.m.f. in the coil at any instant following closure of the switch
(e) Determine the natural frequency of the circuit in cycles per second
(f) evaluate the energy stored in the capacitor at the instant of switching and at the instant $0{\cdot}2 \times 10^{-3}$ seconds after closure of the switch.

2 The circuit of question 1 now includes a resistor of value 50 ohms. Evaluate

(a) the damping ratio ζ

(b) the damped frequency in cycles per second (see Appendix I)

(c) by how much the resistance value must be increased if the circuit is to be critically damped.

4

SINUSOIDAL VARIATIONS

4.1 Frequency and phase

Reference has been made, in dealing with oscillatory systems, to quantities which vary sinusoidally with time at a certain frequency. This form of variation is extremely common; the vast majority of domestic and industrial supply voltages, for example, have a waveform which can be expressed in mathematical terms as

$$v = V_m \sin \omega t$$

where ω is a constant known as the angular frequency of the voltage wave. (Figure 4.1.) From the diagram it can be seen that the voltage reaches its maximum value when the time t_1 is such that

$$\omega t_1 = \pi/2$$

or when $$t_1 = \pi/2\omega.$$

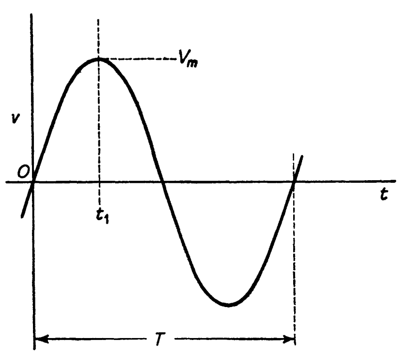


FIGURE 4.1

This represents one quarter of a complete cycle. The time of one cycle is therefore given by

$$T = 4t_1 = 2\pi/\omega.$$

Time, for this purpose, is usually measured in seconds; and it could be said that T is the number of seconds per cycle. It is usual, however,

to refer rather to the number of cycles per second and this is known as the frequency of the waveform.

Thus frequency
$$f = 1/T$$
$$= \omega/2\pi$$
or
$$\omega = 2\pi f.$$

A quantity which varies sinusoidally with time is thus apparently completely defined by its maximum value and its frequency. This is not the case, however, when more than one such quantity is being considered. Suppose, for example, there were two such voltages, represented as A and B; and further suppose that for some reason the waveshape of B, although of the same frequency as that of A, commenced at a different instant of time. A sketch of such a pair of waves is shown in Figure 4.2.

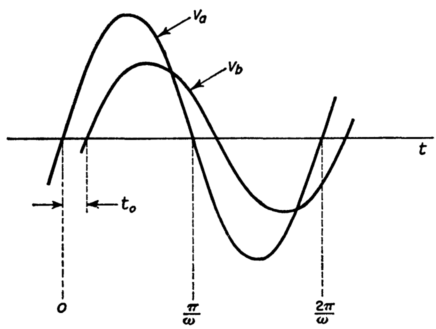


FIGURE 4.2

Waveshape A could be expressed as

$$v_a = V_{am} \sin \omega t$$

and that of B would be

$$v_b = V_{bm} \sin \omega(t - t_0).$$

This indicates that the waveshapes of A and B are similar in form but the value of B is reached at an instant of time t_0 after the corresponding value of A. If A goes through zero, B passes through zero at a time t_0 later, and so on; B is said to be out of phase with A; in this case it is 'lagging' on A. If t_0 were displaced to the opposite side of the origin, B would be 'leading' A. The two waveshapes have been expressed directly in terms of time and angular frequency. A more simple representation is in terms of angle:

$$v_a = V_{am} \sin \beta$$

and

$$v_b = V_{bm} \sin (\beta - \alpha)$$

where $\beta = \omega t$ (the variable) and $\alpha = \omega t_0$ (the constant relating to the phase of B).

Thus it is quite common practice to refer to the phase *angle* between such sinusoidal quantities; but the term should not obscure the fact that it indicates a time difference in the waveshapes. The possibility of a phase difference between two voltages immediately shows that a great deal of care must be exercised when combining the two. If two batteries are connected in series, the voltage measured between the extreme terminals will be the algebraic sum of the two (algebraic because the voltages may be additive or in opposition). If two voltages such as A and B are acting in series, there is in effect a summation at each instant of time and any phase difference must necessarily be taken into account. In introducing this concept of time difference, or phase angle, reference has been made entirely to a pair of voltage waves. However it should be appreciated that the application of a sinusoidal voltage to a circuit may result in a current which also varies sinusoidally with time; whether the voltage and the associated current are in phase depends entirely on the nature of the circuit.

4.2 Resistive circuit

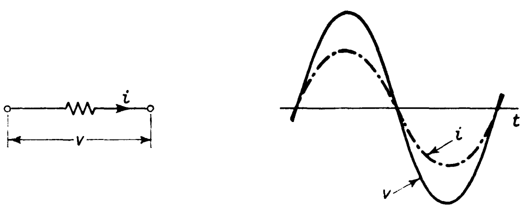


FIGURE 4.3

In Figure 4.3 the circuit comprises only resistance R. Suppose that the current through the resistor is

$$i = I_m \sin \omega t$$

then the instantaneous voltage is given by

$$\begin{aligned} v &= iR \\ &= I_m R \sin \omega t \\ &= V_m \sin \omega t \end{aligned}$$

where $V_m = I_m R$.

4.3 Inductive circuit

In Figure 4.4 only inductance is considered. The applied terminal voltage and the induced e.m.f. are v and e respectively.

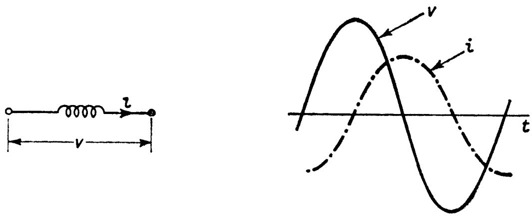


FIGURE 4.4

Hence
$$v + e = 0$$
$$v - L\frac{di}{dt} = 0$$

and if, as before, it be assumed that the current is varying sinusoidally

$$v - L\frac{d}{dt}I_m \sin \omega t = 0$$

or
$$v = \omega L I_m \sin(\omega t + \pi/2)$$
$$= V_m \sin(\omega t + \pi/2)$$

where $V_m = I_m \omega L$.

The voltage across the inductor therefore leads the current by a phase angle of $\pi/2$. In terms of time the two wave shapes are separated by a time difference of $\pi/2\omega$. It is an apparently odd fact that when the voltage is a maximum the current is zero in such a circuit; and vice-versa.

4.4 Capacitive circuit

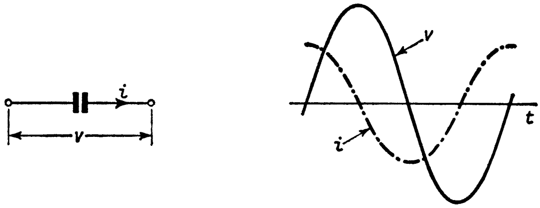


FIGURE 4.5

In Figure 4.5 only capacitance is present in the circuit. Suppose that the voltage across the capacitor is

$$v = V_m \sin \omega t$$

then
$$q = Cv = CV_m \sin \omega t$$

and $$i = \frac{dq}{dt} = \omega C V_m \sin\left(\omega t + \frac{\pi}{2}\right)$$
$$= I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

where $I_m = \omega C V_m$.

In the two previous cases the current has been defined as

$$i = I_m \sin \omega t$$

In order to give correspondence between this capacitive circuit case and the previous two, both current and voltage waves for this case must be shifted back by $\pi/2$. In that event if

$$i = I_m \sin \omega t$$

then $$v = V_m \sin\left(\omega t - \frac{\pi}{2}\right).$$

Again, this phase shift of $\pi/2$ is apparent, but this time the voltage lags on the current. The three separate components give quite different effects when carrying a current which varies sinusoidally with time. In each case there is a relationship between the amplitude of peak voltage and peak current which may be expressed in general terms as

$$I_m = V_m/X.$$

4.5 Reactance

In the first case (Section 4.3) $X = R$ which is the circuit resistance.

In the second case (Section 4.4) $X = \omega L$ or $2\pi f L$.

In the third case (Section 4.5) $X = 1/\omega C$ or $1/2\pi f C$.

These last two terms cannot be called 'resistance' because there is no physical resistance in either circuit. Yet, like resistance, they determine the values of current resulting from the application of a voltage to a circuit. It should be noted that, unlike resistance, they are both functions of frequency. They are termed 'reactance' and it can be seen that inductive reactance increases proportionally with frequency whilst capacitive reactance varies inversely with frequency. Now suppose that all three elements are connected in series (Figure 4.6). The common current in each is

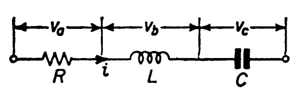


FIGURE 4.6

$$i = I_m \sin \omega t$$

whilst the component voltages are

$$v_a = I_m R \sin \omega t$$

$$v_b = I_m \omega L \sin \left(\omega t + \frac{\pi}{2}\right)$$

$$v_c = \frac{I_m}{\omega C} \sin \left(\omega t - \frac{\pi}{2}\right).$$

The total voltage will be the sum of these: $v_a + v_b + v_c$.

At this stage no attempt will be made to determine a general expression by direct summation of these component terms. This could be done; but it is shown later that there is an extremely simple technique for combining such functions. The preceding sections set out the fundamentals for the study of circuits which are subject to a disturbance varying sinusoidally with time. Before dealing with the means of manipulating these quantities, the precise meaning of terms such as voltage and current and power in such circuits will be considered.

4.6 Mean and root mean square values

There is no particular ambiguity when one expresses the magnitude of a voltage, such as that of a battery, as, say, twelve volts. However, if a voltage is varying with time, the 'magnitude' may have several meanings. For example, it may be the value at a particular instant of time, or the maximum value, or the average value taken over a certain period. If a voltage is expressed as

$$\begin{aligned} v &= V_m \sin \omega t \\ &= V_m \sin \beta \end{aligned}$$

then obviously the average value over a complete cycle is zero. In general the time-average value of any function $y = f(t)$ between instants t_1 and t_2 is

$$y_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)\, \mathrm{d}t$$

then
$$y_{av} = \frac{1}{\pi} \int_0^{\pi} V_m \sin \beta \, \mathrm{d}\beta$$

in the case of one half cycle of the sine wave of voltage. This results in an average value of $2V_m/\pi$. However, this average value over one half of the cycle is of no great importance in circuit calculations. For that matter, the peak value would be more relevant since it represents the extent to which the insulation of the circuit is stressed under normal conditions.

Then what, precisely, do we mean when we say that the supply voltage is 240 volts or an electric fire takes 8 amperes? In fact, these are known as the 'root-mean-square' or r.m.s. values. If the term sounds rather odd, there is a perfectly good reason for its use; once again, integral calculus is used.

Suppose that a direct current I flows in a resistance R. Then the power dissipated in the resistor is

$$P = I^2R.$$

If the current were of the form $i = I_m \sin \beta$, the power at any instant will be given by

$$p = i^2R = I_m^2R \sin^2 \beta.$$

This shows that the power is varying from instant to instant. The average value of power over one complete cycle of current is given by

$$\begin{aligned} P &= \frac{1}{2\pi}\int_0^{2\pi} I_m^2R \sin^2 \beta \, \mathrm{d}\beta \\ &= \frac{I_m^2R}{4\pi}\int_0^{2\pi} (1-\cos 2\beta) \, \mathrm{d}\beta \\ &= \frac{I_m^2R}{4\pi}\left[\beta - \frac{\sin 2\beta}{2}\right]_0^{2\pi} \\ &= \frac{I_m^2R}{2} \quad \text{or} \quad \left(\frac{I_m}{\sqrt{2}}\right)^2 R. \end{aligned}$$

This is the power that would be dissipated by a steady current of $I_m/\sqrt{2}$ in the resistor. It is known as the r.m.s. value of the alternating current. It is the same value which would be obtained by taking the expression for current

$$i = I_m \sin \beta$$

squaring it $\qquad i^2 = I_m^2 \sin^2 \beta$

finding the mean square value over the cycle

$$\begin{aligned} &= \frac{1}{2\pi}\int_0^{2\pi} I_m^2 \sin^2 \beta \, \mathrm{d}\beta \\ &= \frac{I_m^2}{2} \end{aligned}$$

and taking the square root

$$I = \frac{I_m}{\sqrt{2}}.$$

Hence the expression root-mean-square. In the same way the corresponding value for a voltage of maximum value V_m is

$$V = \frac{V_m}{\sqrt{2}}.$$

The magnitude of voltage or current which is normally specified is therefore associated with the direct value which would give the same power dissipation, or heating effect.

4.7 Power

Power has been examined in terms of an 'I^2R' loss. In general, when a constant direct voltage is applied to a resistive circuit, the power is simply

$$P = VI.$$

None of these quantities varies with time; but if the voltage is varying according to the law

$$v = V_m \sin \beta$$

it is a little difficult to imagine precisely what is meant by the power in the circuit. Strictly, we should always think of the instantaneous power as being

$$p = vi.$$

It has been shown that it is possible for the voltage and current to be out of phase. Suppose that a voltage is applied to a circuit

$$v = V_m \sin \beta$$

and that the resulting current is

$$i = I_m \sin (\beta + \varphi)$$

where φ is the phase angle between them, then

$$p = V_m I_m \sin \beta \sin (\beta + \varphi).$$

The instantaneous power thus involves the product of two sine waves. It is a case where certain trigonometrical expansions may be used. In general terms

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

in this case, therefore

$$\sin \beta \sin (\beta + \varphi) = \tfrac{1}{2}[\cos (-\varphi) - \cos (2\beta + \varphi)]$$

hence

$$p = \frac{V_m I_m}{2} \cos \varphi - \frac{V_m I_m}{2} \cos (2\beta + \varphi).$$

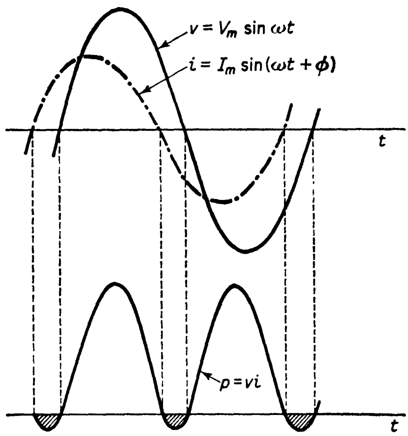


FIGURE 4.7

This is the effect of taking two sine waves, displacing one from the other in phase, and multiplying them ordinate by ordinate. The resultant waveform (Figure 4.7) shows the way in which the instantaneous power varies from instant to instant. It should be noticed that there are periods when the power is negative. That is, the circuit is actually returning power to the source during part of the cycle. A study of the expression for the power shows that it comprises two terms:

(i) A constant term $(V_m I_m/2) \cos \varphi$, since V_m, I_m and φ are all constant quantities.

(ii) A term which varies sinusoidally with time at double the frequency of the voltage and current waves.

The average value of this second component over a complete cycle is zero; in other words the average value of the power is given by the first term and

$$P = \frac{V_m I_m}{2} \cos \varphi$$

$$= VI \cos \varphi$$

where V and I are the r.m.s. values of voltage and current. The ratio given by dividing the power in watts by the product of the r.m.s. voltage and the r.m.s. current in amperes is known as the 'power factor' of the alternating current circuit. That is, the power factor $= P/VI$. It can be seen that in the particular case of sinusoidal variations this power factor is given by $\cos \varphi$.

When we speak of the power in such a circuit we mean this time-average value. It is the energy transfer taking place during a cycle divided by the duration of the cycle. It is clear from the expression for power that if $\varphi = \pi/2$, then the average power is zero. This condition applies to a circuit which is purely inductive or purely capacitive.

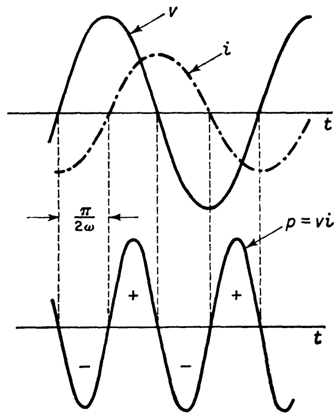


FIGURE 4.8

The waveshapes relating to the case of a purely inductive load are shown in Figure 4.8. The double frequency effect and zero average value are clearly seen in this case which shows that during one period energy is being put into the circuit and during the next is being returned entirely to the source. No energy is dissipated and no useful work is done. The input energy establishes, in this case, a magnetic field; it is the collapse of the field which results in a return of energy to the source.

Exercises

1 Two voltages of peak amplitude 10 volts and 8 volts respectively vary sinusoidally with time at 1000 cycles per second. The second voltage reaches its peak value $0{\cdot}125 \times 10^{-3}$ seconds after the other.

(a) Express the voltages in mathematical form

(b) Determine the phase angle between them.

2 In each of the separate cases shown in Figures 4.3, 4.4 and 4.5 the circuit current is defined as

$$I = 0{\cdot}4 \sin 314t \text{ amperes}$$

Write down expressions for voltage in each case if

(a) $R = 10$ ohms

(b) $L = 0{\cdot}1$ henrys

(c) $C = 8 \times 10^{-6}$ farads.

3 A voltage waveform is defined by the fact that from $t = 0$ to $t = 0{\cdot}05$ seconds it follows the law

$$v = 100t.$$

At $t = 0{\cdot}05$ seconds the voltage falls instantaneously to zero and the waveform repeats. Determine

(a) the peak value

(b) the average value

(c) the r.m.s. value of the voltage wave.

4 A voltage defined by

$$V = 20 \sin 2000\pi t$$

exists across a circuit. The resulting sinusoidal current has a peak value of $1{\cdot}0$ ampere, reached $0{\cdot}166 \times 10^{-3}$ seconds after that of the voltage wave.

(a) Derive an expression for the instantaneous power.

(b) Evaluate the mean power.

(c) Determine the value of the power factor.

5

THE USE OF VECTORS

5.1 Introduction

A brief reference was made (Section 4.5) to the fact that there is a special technique for manipulating quantities which vary sinusoidally with time. It makes use of the concept that such a quantity may be completely represented by a vector rotating at a constant speed. The diagram (Figure 5.1) shows a line of length V_m rotating at an angular speed ω radians per second. If time t is measured from the datum $X'X$ the projection of the line on to the vertical axis $Y'Y$ at any instant will be $V_m \sin \omega t$. If V_m represents the maximum amplitude of a voltage wave it follows that the value of voltage at any instant will be completely defined by the position of the line. The whole wave-shape may be developed from a sequence of lines spaced by equal time intervals (Figure 5.2). Developing this idea further, let us consider two sine waves displaced in time

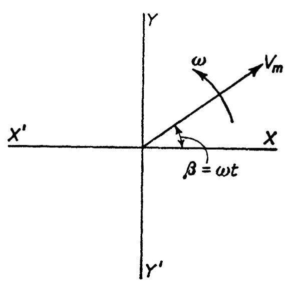


FIGURE 5.1

$$v_1 = V_{1m} \sin \omega t \qquad = V_{1m} \sin \beta$$

and

$$v_2 = V_{2m} \sin \omega(t - t_0) = V_{2m} \sin (\beta - \alpha).$$

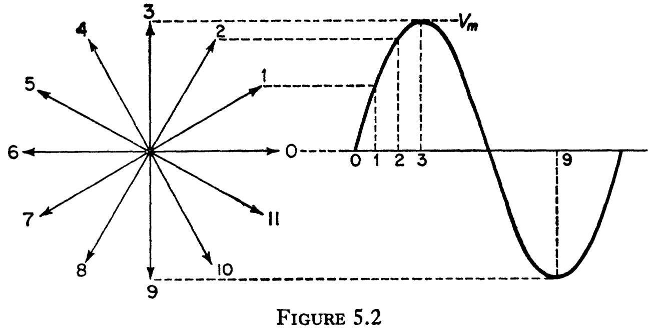


FIGURE 5.2

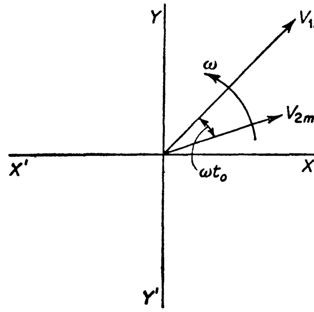


FIGURE 5.3

These may be represented on the same diagram (Figure 5.3). The direction of rotation has been represented as anticlockwise; this is the standard convention. Of the lines V_1 and V_2, V_1 passes through any point in the plane of the axes first, followed by V_2; that is, V_1 leads V_2 by a time interval t_0 or a phase angle α. Of course one might equally say that V_2 lags V_1 by a phase angle α, or even that it leads it by an angle of $(2\pi - \alpha)$.

In general, the concept of a rotating line is associated with instantaneous values of the sine wave. Any consideration requiring a knowledge of the value of a quantity at a particular instant of time must necessarily be referred back to the mathematical expression, the graphical plot, or the rotating line. However, we may only be concerned with the relative amplitude and phase of a number of sinusoidal quantities. In this event a number of stationary lines of the correct length and relative phase angle would summarise completely all the required information. These lines are usually referred to as 'vectors' although there are objections to the use of this term since, from the mathematical viewpoint, it has a somewhat wider meaning. For this reason the terms 'phasor' or 'complexor' are sometimes used for this particular method of representing simple harmonic motion.

It is usual to specify that one such vector in a group should lie along the X'X axis. This is termed the 'vector of reference' since all others are referred to it. Thus in the diagram (Figure 5.4) V_1 is the vector reference and V_2, V_3 and V_4 have phase angles of α_2, α_3, α_4 respectively—lagging or leading. The actual length of the vector could be the maximum value of the quantity, in which case projection on to the vertical as the system rotates about the axis would give instantaneous values. Alternatively the lengths can be taken to represent the r.m.s. values of the various quantities, and this is the usual practice.

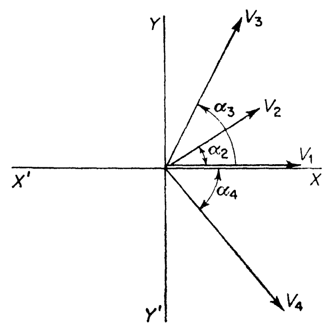


FIGURE 5.4

As the concept of a set of rotating lines is dropped and the vectors are represented not in terms of maximum amplitude but in terms of the r.m.s. value, so the whole idea that these lines represent quantities which vary sinusoidally with time tends to become more remote. It cannot be stressed too greatly that, in this context, vectors are a very convenient shorthand way of manipulating such quantities. The phase angle tends to be regarded as a phase angle between two lines; in fact it should always register as a measure of the time displacement between two waveshapes.

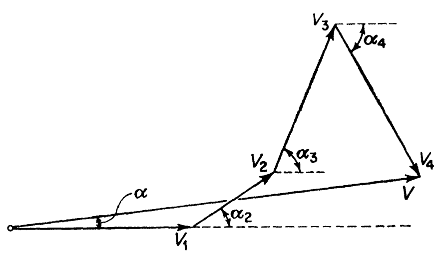


FIGURE 5.5

The vectors have, up to now, been shown radiating from the origin. Summation of such a set is a simpler process if they are made to continue onwards as shown in the diagram (Figure 5.5). The point of V_1 is the origin of V_2 and so on. The resultant vector is represented by a straight line from the true origin to the end point. Using the plus sign to indicate vector addition we may write

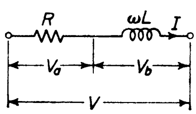
$$\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3 + \bar{V}_4.$$

The resultant is thus completely defined in magnitude and phase relative to the vector of reference. The application of this technique to particular circuits is a subject in itself. Quite obviously each individual circuit may be represented by its own particular set of vectors. In the sections which follow, series combinations of resistance, inductance and capacitance will be considered.

5.2 Series resistance and inductive reactance

The current through the two circuit elements (Figure 5.6) is assumed to be

$$i = I_m \sin \omega t.$$



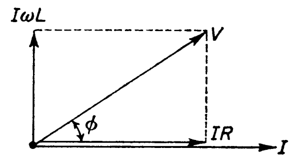


FIGURE 5.6

Since it is common to both elements it is convenient to make it the quantity to which all others are referred. It then follows that

$$\begin{aligned} v_a &= V_{am} \sin \omega t &&= I_m R \sin \omega t \\ v_b &= V_{bm} \sin (\omega t + \pi/2) &&= I_m \omega L \sin (\omega t + \pi/2). \end{aligned}$$

In terms of r.m.s. quantities these may conveniently be represented as

$$\begin{aligned} V_a &= IR && \text{in phase with current} \\ V_b &= I\omega L && \text{leading the current by } \pi/2. \end{aligned}$$

From the diagram, if the total voltage is V,

$$V^2 = (IR)^2 + (I\omega L)^2$$

or

$$\frac{V}{I} = [R^2 + (\omega L)^2]^{1/2}$$

and

$$\varphi = \tan^{-1} \frac{I\omega L}{IR}$$

$$= \tan^{-1} \frac{\omega L}{R}.$$

Thus the magnitude and phase of the voltage and current will be dependent upon a combination of the values of resistance and inductive reactance.

5.3 Series resistance and capacitance

In this case (Figure 5.7) and working directly in terms of r.m.s. values

$$V_a = IR \quad \text{in phase with the current}$$
$$V_c = I/\omega C \quad \text{lagging the current by } \pi/2.$$

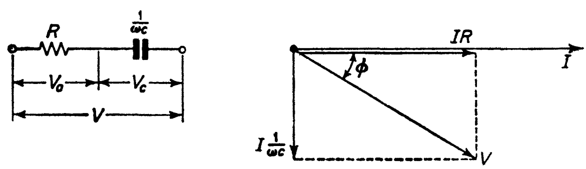


FIGURE 5.7

The associated relationships between total voltage and current are

$$V^2 = (IR)^2 + \left(\frac{I}{\omega C}\right)^2$$

or

$$\frac{V}{I} = \left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]^{1/2}$$

and

$$\varphi = \tan^{-1} \frac{I/\omega C}{IR}$$
$$= \tan^{-1} \frac{1}{\omega CR}.$$

5.4 Impedance

In a resistive circuit the current is limited by the value of resistance. In a circuit containing only inductance or only capacitance we have seen that, depending on the frequency, the current is limited by what was termed the reactance. Now we have cases where V/I is a combination of resistance and reactance. In each case the circuit is said to offer a certain *impedance* to the source. In the two cases we have examined we could have related the impedance, the resistance and the reactance by writing

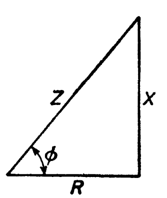


FIGURE 5.8

$$Z^2 = R^2 + X^2.$$

This gives rise to the 'impedance' triangle shown in Figure 5.8, in which it can be seen that

$$R = Z \cos \varphi$$

and

$$X = Z \sin \varphi.$$

5.5 General series circuit

Returning once again to the circuit which comprises resistance, inductance and capacitance (Figure 5.9) it can be seen that in terms of r.m.s. quantities

$V_a = IR$ in phase with the current
$V_b = I\omega L$ leading the current by $\pi/2$
$V_c = I/\omega C$ lagging the current by $\pi/2$.

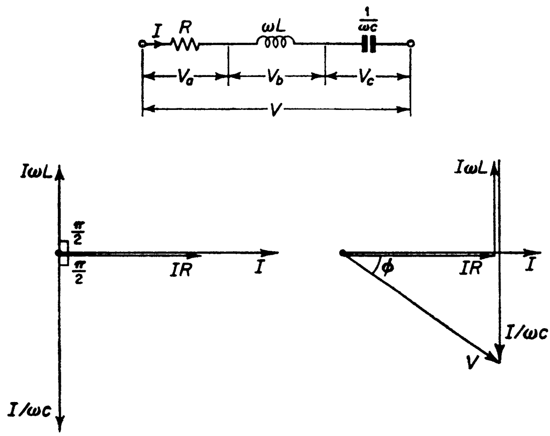


FIGURE 5.9

The vector addition shows that

$$V^2 = (IR)^2 + \left(I\omega L - \frac{I}{\omega C}\right)^2$$

or

$$\frac{V}{I} = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}.$$

φ, the phase angle by which the voltage lags on the current is given by

$$\tan^{-1}\frac{\omega L - 1/\omega C}{R}.$$

Certain deductions can be made from these general expressions:

(i) If $\omega L > 1/\omega C$, then φ is a positive angle and the voltage leads the current.

(ii) If $\omega L < 1/\omega C$, then φ is a negative angle and the voltage lags on the current. This is the case shown in the diagram. It follows that the behaviour of the circuit is dependent on which of the reactances, inductive or capacitive, is predominant.

(iii) The impedance which limits the circuit current is given by

$$Z = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}.$$

(iv) The circuit is general in that all three elements are included. The behaviour of a circuit containing only resistance and inductance could be deduced by putting $1/\omega C = 0$. Similarly if $\omega L = 0$ the relationships are those for a circuit containing resistance and capacitance only.

5.6 Resonance

Of particular significance is the case where the two reactance values are equal; that is, where $\omega L = 1/\omega C$. In this event the current is given by $I = V/R$, and the two equal reactance voltages by

$$V_b = I\omega L = V\frac{\omega L}{R}$$

$$V_c = \frac{I}{\omega C} = V\frac{1}{\omega CR}.$$

Since (ωL) and $(1/\omega C)$ are both quite independent of the resistance R, it follows that by choosing circuit values such that for this condition resistance is small compared with reactance, the voltage of the inductor or capacitor may be many times greater than that impressed on the circuit. The condition will arise by virtue of the fact that frequency, inductance and capacitance are related through

$$\omega L = \frac{1}{\omega C}$$

i.e.
$$\omega = \sqrt{\left(\frac{1}{LC}\right)}$$

or
$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}\right)}.$$

The frequency of the voltage impressed on the circuit is the same as the natural frequency of the circuit. It is a common enough phenomenon for small disturbances in one part of a system to give rise to a bigger disturbance elsewhere. This is a case in point: if, for example, the magnitude of the input is 1 volt and the equal inductive and capacitive reactances are ten times the circuit resistance, then the voltage across either of the separate reactances will be 10 volts.

5.7 The general series circuit, instantaneous conditions

The preceding sections were concerned with a technique for finding, comparatively readily, the properties of a particular circuit. As with all techniques, it tends to mask the overall behaviour of the circuit. We were intent upon finding out as much as we could about the steady state behaviour of circuits under conditions of sinusoidal disturbance. But this should not completely satisfy the real seeker after the truth; he would tend to look at the circuit in the following way (Figure 5.10).

At any instant, the voltage across the resistor is iR and across the capacitor is q/C, and the self-induced e.m.f. in the inductor is $-L(di/dt)$. Since the total instantaneous voltage is v

$$v - L\frac{\mathrm{d}i}{\mathrm{d}t} = iR + \frac{q}{C}$$

or

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = v \tag{1}$$

$$= V_m \sin \omega t.$$

This form of equation was discussed earlier when the effects of closing such a circuit on itself were investigated. In that case the impetus given to the circuit came by virtue of the fact that the capacitor was initially charged. In this case the oscillation is maintained by the external disturbance.

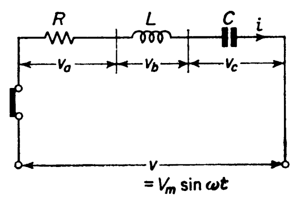


FIGURE 5.10

Undoubtedly the equation for capacitor charge could be solved and the instantaneous current deduced from the fact that $i = \mathrm{d}q/\mathrm{d}t$. We should need to know at what instant of time the external voltage was switched on to the circuit and whether there was any energy stored in the capacitor or inductor at the instant of switching. Given these initial conditions however, a general expression for the charge at any time could be deduced by solving the differential equation. We shall find ourselves faced with an important question; but first let us review the situation. We have already deduced that the current in the circuit is lagging on the voltage by an angle φ equal to $\tan^{-1}(\omega L - 1/\omega C)/R$ and given by the expression

$$I = \frac{V}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}}.$$

If this were to be expressed in terms of instantaneous value of current then since

$$I_m = \sqrt{(2)}I$$

and

$$V_m = \sqrt{(2)}V$$

the current

$$i = \frac{V_m}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \sin(\omega t - \varphi)$$

and charge ($q = \int i \, \mathrm{d}t$)

$$q = -\frac{V_m}{\omega[R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \cos(\omega t - \varphi).$$

Does, then, this expression for q satisfy completely the differential equation relating charge and voltage? The operative word here is 'completely'; this expression for instantaneous charge certainly satisfies the differential equation. It is a useful, if somewhat lengthy, exercise to check this by substituting for q and its derivatives in the left hand side of Equation (1). The answer reduces to $V_m \sin \omega t$. The fact is, however, that this solution is only part of the complete answer.

Suppose that

$$q_1 = F_1(t)$$

where $F_1(t)$, some function of time, is a solution of the equation

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0$$

that is, the same equation but with zero applied voltage; and suppose that the particular solution which we have derived (for the case where there *is* an applied voltage) is represented by

$$q_2 = F_2(t).$$

Then the complete solution of

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = V_m \sin \omega t$$

is given by

$$\begin{aligned} q &= q_1 + q_2 \\ &= F_1(t) + F_2(t). \end{aligned}$$

This may be verified by substitution into the differential equation which then reduces to

$$\left(L\frac{\mathrm{d}^2q_1}{\mathrm{d}t^2} + R\frac{\mathrm{d}q_1}{\mathrm{d}t} + \frac{q_1}{C}\right) + \left(L\frac{\mathrm{d}^2q_2}{\mathrm{d}t^2} + R\frac{\mathrm{d}q_2}{\mathrm{d}t} + \frac{q_2}{C}\right) = V_m \sin \omega t.$$

It has already been assumed that part of the solution afforded by q_1 is such that the left-hand bracketed term is zero. It follows that a complete solution of the equation must involve

(i) The answer obtained by solving

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

and

(ii) The particular expression for q which, substituted in the left hand side of the equation, results in the steady disturbance term $V_m \sin \omega t$.

The first term takes account of the transient effect which would be apparent when the circuit is first switched on. This is dependent on the instant of time at which the input voltage is applied. If the circuit is switched on at an instant when the voltage wave is passing through zero the transient takes one form. Should switching occur at the instant of maximum applied voltage the transient would be quite different.

A complete solution would also show that the transient effects die away with time and that eventually the circuit 'settles down' to a steady state mode. It is this steady state behaviour given by the second part of the solution of the differential equation with which we have been dealing in applying vector methods.

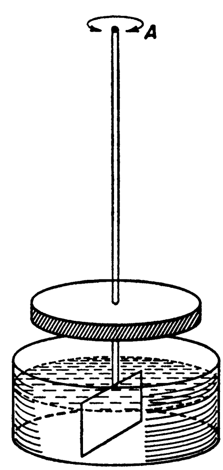


FIGURE 5.11

5.8 The torsional pendulum

Writing down the complete equation of motion of the system is an exercise which could equally be applied to the torsional pendulum. Consider the pliant rod with a massive disc attached and with some form of viscous damping present (Figure 5.11). Suppose that the upper end of the rod is twisted to and fro at an angular frequency ω. If at any instant of time the angle of twist at point A is given by $\theta_1 \sin \omega t$, and if at this instant the angle through which the disc has turned is θ, the torque exerted on the disc is

$$T = \frac{\theta_1 \sin \omega t - \theta}{\lambda}$$

where λ is the compliance of the rod. This is diminished by the viscous

friction torque which is proportional to the angular speed of the disc. It follows that the complete torque equation will be given by

$$\frac{\theta_1 \sin \omega t - \theta}{\lambda} - K\frac{d\theta}{dt} = J\frac{d^2\theta}{dt^2}$$

or

$$J\frac{d^2\theta}{dt^2} + K\frac{d\theta}{dt} + \frac{\theta}{\lambda} = \frac{\theta_1}{\lambda}\sin \omega t.$$

This is precisely the form of equation as that derived for the network. A complete analogy between quantities has already been set out, the only addition being the correspondence between voltage and the term θ_1/λ which is, in fact, torque. Now from the expression for instantaneous charge in the general series circuit it will be seen that the maximum charge is given by

$$\frac{V_m}{\omega[R^2 + (\omega L - 1/\omega C)^2]^{1/2}}.$$

Making use of the analogy between mechanical and electrical systems it can be seen that the maximum angular displacement of the disc is given by

$$\theta_m = \frac{\theta_1}{\omega\lambda[K^2 + (\omega J - 1/\omega\lambda)^2]^{1/2}}.$$

It can be seen that this maximum 'swing' is not only dependent upon the system constants but also on the frequency of the disturbance.

This relationship may be modified to the form

$$\frac{\theta_m}{\theta_1} = \frac{1}{\omega\lambda J[K^2/J^2 + (\omega - 1/\omega\lambda J)^2]^{1/2}}$$

and the parameters $\omega_n^2 = 1/\lambda J$ and ζ defined by $2\zeta\omega_n = K/J$ introduced. This substitution results in an expression

$$\frac{\theta_m}{\theta_1} = \frac{\omega_n^2}{[(\omega^2 - \omega_n^2)^2 + (2\zeta\omega_n\omega)^2]^{1/2}}.$$

Examination of this relationship shows that if $\zeta = 1/\sqrt{2}$, θ_m/θ_1 can never be greater than unity; in other words, whatever the frequency of the disturbance, the maximum swing of the disc can never exceed that of the disturbance. The significance of this particular value of ζ is shown by plotting θ_m/θ_1 for various values of ζ over a range of angular frequencies. Figure 5.12a shows this family of curves from which it can be seen that for values of ζ less than $1/\sqrt{2}$ the maximum amplitude of oscillation of the disc may well be greater than that of the disturbance. By differentiating the expression for θ_m/θ_1 with respect to ω and equating to zero it can be shown that the peak of the maximum

swing curve for this condition occurs when $\omega = \omega_n(1 - 2\zeta^2)^{1/2}$. At this value of angular frequency

$$\frac{\theta_m}{\theta_1} = \frac{1}{2\zeta(1 - \zeta^2)^{1/2}} \qquad \text{(Figure 5.12b)}$$

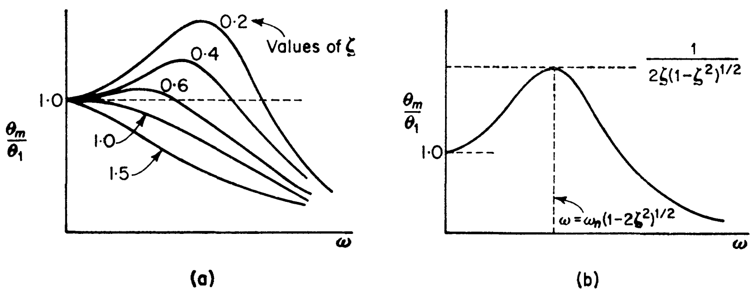


FIGURE 5.12

These curves are plotted to a base of angular frequency, but it should be appreciated that since $\omega = 2\pi f$ the abscissa could readily be calibrated in terms of frequency in cycles per second.

At low frequencies the maximum angular displacement is comparable with that of the input. Under these conditions the disc attempts to follow the input disturbance. At the upper end of the frequency range the disc displacement becomes infinitesimally small; it just cannot respond to rapid variations of the input. In between these two extremes, provided that the conditions set out above are met, there is a peak in the characteristic at a particular value of frequency. The greater the damping ratio, the less pronounced will be the peak; and if this ratio should be greater than $1/\sqrt{2}$ there will be no peak at all.

This departure from the study of a circuit to the study of a mechanical system reinforces the idea expressed earlier that the behaviour of quite different systems may be studied together provided that they are described by the same mathematical form of equation. We have all heard of singers with voices so powerful that they make the cups rattle. We might say that this is a case where a note which is a mixture of frequencies sets up comparatively small variations of air pressure. These transmit small disturbances to devices sufficiently sensitive to respond to a particular frequency. Hence the rattle. In the same way a voltage wave may consist of a mixture of frequencies; and a circuit which is 'tuned' to one particular value of frequency will accept the component voltage of that frequency and reject, in some measure, the remainder.

5.9 Connexion of impedances in series

It has been shown that an impedance is made up of two terms

$$R = Z \cos \varphi$$

and
$$X = Z \sin \varphi.$$

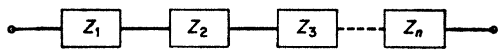


FIGURE 5.13

If a circuit comprises a series of such impedances (Figure 5.13) then the total resistance will be given by

$$R = Z_1 \cos \varphi_1 + Z_2 \cos \varphi_2 + \cdots + Z_n \cos \varphi_n$$

and the total reactance by

$$X = Z_1 \sin \varphi_1 + Z_2 \sin \varphi_2 + \cdots + Z_n \sin \varphi_n$$

due account being taken of the sign of the angle. Thus the total impedance is given by

$$Z = (R^2 + X^2)^{1/2}$$

and the resultant phase angle by

$$\varphi = \tan^{-1} X/R.$$

5.10 Parallel connexion

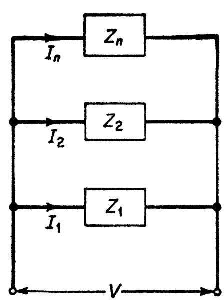


FIGURE 5.14

A number of impedances are connected in parallel as shown in Figure 5.14. This is a case where it is appropriate to take the voltage as the vector of reference, since it is common to all branches. The currents in the branches will be

$$I_1 = V/Z_1 \quad \text{at phase angle } \varphi_1$$
$$I_2 = V/Z_2 \quad \text{at phase angle } \varphi_2$$

and in general

$$I_n = V/Z_n \quad \text{at phase angle } \varphi_n.$$

These are represented on a vector diagram in Figure 5.15.

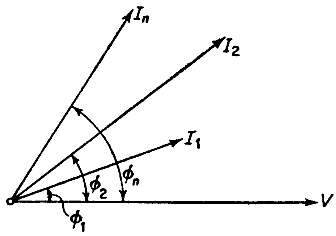


FIGURE 5.15

The total component of current in phase with the voltage is

$$I_P = I_1 \cos \varphi_1 + I_2 \cos \varphi_2 + \cdots + I_n \cos \varphi_n$$

and the total current in phase 'quadrature' with the voltage, that is, lagging or leading it by $\pi/2$, is

$$I_Q = I_1 \sin \varphi_1 + I_2 \sin \varphi_2 + \cdots I_n \sin \varphi_n$$

due account again being taken of the sign of the phase angle.

The total current is then given by

$$I = (I_P^2 + I_Q^2)^{1/2}$$

and the phase angle by

$$\varphi = \tan^{-1} \frac{I_Q}{I_P}.$$

The great advantage of the vector method is that it enables resolution of voltages, in the case of the series circuit, and currents, in the case of the parallel circuit, to give fairly readily the overall properties of a circuit.

5.11 Series-parallel connexion

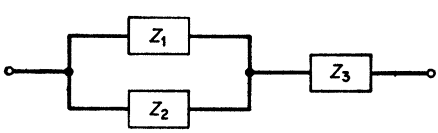


FIGURE 5.16

Now let us consider a network which is a combination of these two cases (Figure 5.16). It is a fact that the impedance which such a circuit presents to a sinusoidal voltage can be determined by combining the two methods outlined above; but this leads to a somewhat unwieldy form of solution. It would, of course, be much simpler if the impedances could be manipulated in much the same way as resistances; the problem is that an impedance represents a relationship between a voltage and a current which are out of phase and any such manipulation must, of necessity, take account of this.

This is where the use of complex notation is of such importance in electrical network theory. The following chapter outlines the use of this method.

Exercises

1 Three voltages

$$v_1 = 10 \sin \omega t$$
$$v_2 = 8 \sin (\omega t + \pi/4)$$
$$v_3 = 6 \sin (\omega t + \pi/6)$$

act in series. By making a vector summation determine

(a) an expression for the instantaneous value
(b) the peak value
(c) the r.m.s. value

of the resultant voltage

2 See Figure 5.6

If
$$R = 300 \text{ ohms}$$
$$L = 0{\cdot}1 \text{ henrys}$$
$$f = 1000 \text{ cycles per second}$$
$$V = 10 \text{ volts r.m.s.}$$

Evaluate

(a) the inductive reactance
(b) the impedance
(c) the current (r.m.s.)
(d) the circuit phase angle
(e) the time-phase difference between voltage and current
(f) the resistance voltage (r.m.s.)

(g) the reactance voltage (r.m.s.)
(h) the average power dissipated.

3 See Figure 5.10

If the r.m.s. voltage is 10 volts and

$$L = 0{\cdot}01 \text{ henrys}$$
$$R = 10 \text{ ohms}$$
$$f = 1000 \text{ cycles per second}$$

(a) determine the value of C which will give resonant conditions.
For this case evaluate

(b) the circuit current (r.m.s.)
(c) the resistance voltage (r.m.s.)
(d) the voltages across the inductive and capacitive elements (r.m.s.)

The frequency is now changed to 500 cycles per second. Determine

(e) the circuit current (r.m.s.)
(f) the resistance voltage (r.m.s.)
(g) the voltages across the inductive and capacitive elements (r.m.s.)
(h) the circuit phase angle.

4 See Figure 5.14

Two circuits are connected in parallel:

Z_1 comprises resistance $R_1 = 10$ ohms and inductive reactance $X_1 = 20$ ohms
Z_2 comprises resistance $R_2 = 9$ ohms and inductive reactance $X_2 = 12$ ohms
The r.m.s. input voltage $= 20$ volts.

Evaluate

(a) the total in-phase component of current (r.m.s.)
(b) the total quadrature component of current (r.m.s.)
(c) the total input current (r.m.s.)
(d) the total phase angle
(e) the average power input.

5 See Figure 5.14

Two circuits are connected in parallel:

Z_1 comprises $R_1 = 10$ ohms and inductance $L = 8 \times 10^{-3}$ henrys
Z_2 is a variable capacitor

The input voltage $= 10$ volts

If the supply frequency is 400 cycles per second determine

(a) the value of capacitance which will result in an input current to the parallel combination which is in phase with the applied voltage

(b) the value of this current (r.m.s.)

Since the input current is in phase with the input voltage the circuit under these conditions behaves as a pure resistance. Show, in general terms, that the effective value of this resistance is given by L/CR and

(c) determine its value for this case.

6

COMPLEX NOTATION

6.1 Introduction

The operator $i = \sqrt{(-1)}$ is a useful mathematical concept which finds application in alternating current circuit theory. However, in the study of networks the symbol i is already much in demand and usually denotes instantaneous current. In order to avoid confusion, therefore, the notation for $\sqrt{(-1)}$ will henceforth be j and not i.

There are two forms of representation of a quantity such as OP (Figure 6.1). The Cartesian or (x, y) form

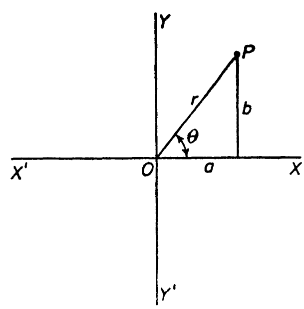


FIGURE 6.1

$$OP = a + jb$$

which means a step of a units along the x or real axis together with a step of b units along the y or imaginary axis. This form is most useful where addition and subtraction of quantities such as OP is necessary.

An alternative method of representation is the polar form

$$OP = re^{j\theta}$$

which is used where multiplication and division is necessary. This second relationship follows from the fact that

$$a = r \cos \theta$$
$$b = r \sin \theta$$

hence $$OP = r(\cos \theta + j \sin \theta).$$

A study of the expansions of $\cos \theta$ and $\sin \theta$ shows that the right hand side of this equation is, in fact, the expansion of $re^{j\theta}$. For simplicity this is usually referred to as $r\underline{/\theta}$. If two such quantities are to be considered

$$a_1 + jb_1 = r_1 e^{j\theta_1}$$
$$a_2 + jb_2 = r_2 e^{j\theta_2}$$

then their product is

$$r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

or

$$r_1 r_2 \underline{/\theta_1 + \theta_2}\,.$$

This is a simpler approach than that which would have to be adopted were the quantities retained in Cartesian form. On the other hand

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

and

$$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2).$$

In these operations the Cartesian form is therefore used. Since it may be necessary to add voltages or currents which vary sinusoidally with time, the complex notation follows naturally from the concept of vector notation.

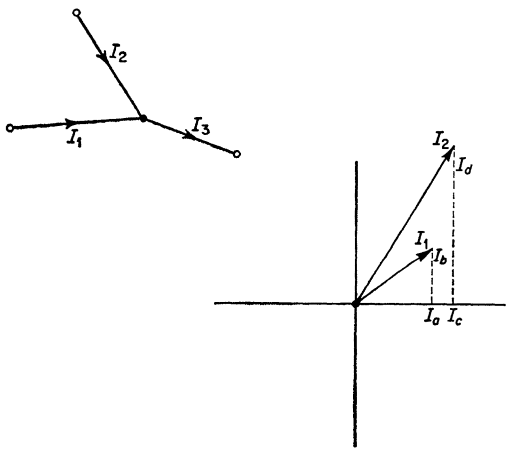


FIGURE 6.2

At a point in a network, for example (Figure 6.2) I_1 and I_2 represent the r.m.s. values of the currents entering the point and I_3 the r.m.s. value of the current leaving it. These may be expressed as vectors with respect to some arbitrary vector of reference.

$$\bar{I}_1 = I_a + jI_b$$
$$\bar{I}_2 = I_c + jI_d.$$

This is the notation of complex numbers but the reader should notice that we continue to use the term *component* for I_a and I_b, as in vector theory.

It is not permissible to write $(I_1 + I_2)$ because this would indicate an arithmetic summation of two current magnitudes, which would, in this context, be meaningless. Currents (and voltages) may only be

added if due account is taken not only of magnitude but also of phase. Summation therefore involves a vector equation

$$\begin{aligned} \bar{I}_3 &= \bar{I}_1 + \bar{I}_2 \\ &= (I_a + I_c) + j(I_b + I_d). \end{aligned}$$

That is, we can add the corresponding components of $\bar{I}_1$ and $\bar{I}_2$ to obtain each component of $\bar{I}_3$.

This would involve exactly the same trigonometrical manipulation as the previous approach making direct use of the vector diagram. After all, the components I_a, I_b, I_c and I_d have to be determined. On this evidence, therefore, there does not seem to be any great advantage in the use of this notation.

6.2 Impedance

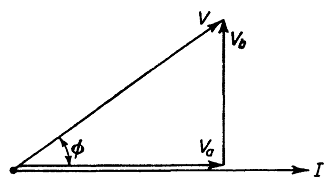


FIGURE 6.3

Suppose that as a result of applying a sinusoidal voltage to a circuit a current I is established and further suppose that the current is represented as the vector of reference (Figure 6.3); that is

$$\bar{I} = I + j0$$

and that the voltage is given by

$$\bar{V} = V_a + jV_b$$

then

$$\begin{aligned} \frac{\bar{V}}{\bar{I}} &= \frac{V_a + jV_b}{I} \\ &= \frac{V_a}{I} + j\frac{V_b}{I} \\ &= \frac{V}{I}\cos\varphi + j\frac{V}{I}\sin\varphi \\ &= Z\cos\varphi + jZ\sin\varphi \quad \text{(and from Section 5.4)} \\ &= R + jX. \end{aligned}$$

In other words not only can voltage and current be represented as

vector quantities, but circuit impedance may also be represented in complex form. It should be noted that φ was taken as a positive angle corresponding to the case of inductive reactance. In the event of capacitive reactance the angle would have been negative. In general, therefore

$$Z = R \pm jX$$

where the '+' sign is taken for inductive reactance and the '−' sign for capacitive reactance. This approach indicates that linear circuits subjected to a sustained sinusoidal disturbance may be solved by reducing them to a single equivalent impedance. In this respect they present no greater problem than that offered by resistance networks, although the fact that complex quantities are used means that the reduction will be somewhat more tedious. That is why it is so important for interchange between Cartesian and polar form to be made freely wherever it is appropriate.

6.3 Example

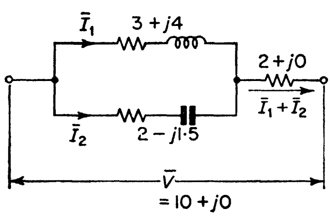


FIGURE 6.4

Figure 6.4 shows a series parallel combination of impedances. The reduction process in this case would be as follows:

In polar form $\quad 3 + j4 = 5 \angle 53° \, 8'$

and $\quad 2 - j1{\cdot}5 = 2{\cdot}5 \angle -36° \, 52'$

hence for the parallel circuit (treating this as one would treat parallel resistors but now using complex values)

$$Z = \frac{\bar{Z}_A \bar{Z}_B}{\bar{Z}_A + \bar{Z}_B}$$

$$= \frac{5 \angle 53° \, 8' \times 2{\cdot}5 \angle -36° \, 52'}{(3 + j4) + (2 - j1{\cdot}5)}$$

$$= \frac{12{\cdot}5 \angle 16° \, 16'}{5 + j2{\cdot}5}.$$

The denominator is now expressed in polar form, giving

$$
\begin{aligned}
Z &= \frac{12{\cdot}5 \underline{/16^\circ\ 16'}}{5{\cdot}59 \underline{/26^\circ\ 34'}} \\
&= 2{\cdot}24 \underline{/-10^\circ\ 18'} \\
&= 2{\cdot}21 - j0{\cdot}4.
\end{aligned}
$$

This polar form has been obtained in order to combine the effective parallel impedance with the series impedance (in this case pure resistance), $2 + j0$.

Thus total impedance

$$
\begin{aligned}
&= (2{\cdot}21 - j0{\cdot}4) + (2 + j0) \\
&= 4{\cdot}21 - j0{\cdot}4.
\end{aligned}
$$

All impedance values are in ohms, and the resulting input current, in amperes, is given by

$$
\begin{aligned}
I &= \frac{10 + j0}{4{\cdot}21 - j0{\cdot}4} \\
&= \frac{10 \underline{/0^\circ}}{4{\cdot}23 \underline{/-5^\circ\ 24'}} \\
&= 2{\cdot}36 \underline{/+5^\circ\ 24'}.
\end{aligned}
$$

Note that the input voltage has been taken as the vector of reference and the answer indicates that the total input current of 2·36 amperes will be leading the voltage by a phase angle of 5° 24′.

There is no reason why a direct approach, provided by setting out the circuit equations, should not be adopted. Taking the circuit through the supply, the top branch of the parallel arrangement and the series resistor

$$
\begin{aligned}
10 + j0 &= \bar{I}_1(3 + j4) + (\bar{I}_1 + \bar{I}_2)(2 + j0) \\
&= \bar{I}_1(5 + j4) + \bar{I}_2(2 + j0).
\end{aligned}
$$

Whilst, since there is a common voltage across the two branches of the parallel circuit

$$\bar{I}_1(3 + j4) = \bar{I}_2(2 - j1{\cdot}5)$$

or

$$0 = \bar{I}_1(3 + j4) - \bar{I}_2(2 - j1{\cdot}5).$$

This involves the solution of two simultaneous equations in which there are complex coefficients. The currents $\bar{I}_1$ and $\bar{I}_2$ and the total current will be complex in form and expressed with respect to the original vector of reference, V. It is doubtful whether this approach offers a quicker solution than the previous method of circuit reduction.

It has the advantage that, in addition to the final solution, the relationships between voltage and current in each part of the circuit may be readily determined.

This problem illustrates a technique, based on sound mathematical principles, which is simple to apply. As with vector methods generally, it tends to take us a further step away from the original concept of sinusoidal variations. But it should never be forgotten that this technique can only apply to circuits which are subjected to sinusoidal variations of voltage and current; circuits where the application of a voltage varying sinusoidally with time results in a sinusoidal current. That is why in any detailed treatment of the subject it is usual to stress the fundamental considerations and not to introduce complex notation until its limitations, as well as its use, can be fully appreciated.

Exercises

1 The voltage across a circuit is given by

$$v = 20 \sin \omega t \quad \text{volts}$$

and the current into the circuit by

$$i = 2 \sin (\omega t - \pi/6) \quad \text{amperes.}$$

Taking the voltage as vector of reference express these as complex r.m.s. quantities and evaluate the impedance of the circuit in complex form.

2 See Figure 5.16

Use the circuit shown in the diagram and take the upper branch impedance as

$$Z_1 = 5 + j0 \quad \text{ohms}$$

the lower branch impedance as

$$Z_2 = 4 + j5 \quad \text{ohms}$$

and the series element as

$$Z_3 = 3 - j4 \quad \text{ohms.}$$

A current of $5 + j0$ amperes flows in the lower branch. Evaluate

(a) the voltage across parallel elements (r.m.s.)
(b) the current in the upper branch (r.m.s.)
(c) the total circuit current (r.m.s.)
(d) the voltage across the series element (r.m.s.)
(e) the total circuit voltage (r.m.s.)
(f) the power dissipated in each resistive element
(g) the total power dissipated
(h) the equivalent series impedance of the whole circuit.

7

PRACTICAL APPLICATIONS

7.1 Introduction

A serious study of any of the specialist subjects in the field of electrical engineering requires a thorough knowledge of circuit theory and the associated mathematics. These specialist studies are concerned with the design and performance of electrical machines, the interconnexion of power supply systems, the design and use of electronic devices, telecommunication systems, special methods of measurement, and the design of control systems which meet a certain requirement.

In endeavouring to pick out particular applications there is therefore no shortage of possible examples. Each device will have associated with it a so-called 'equivalent' circuit to which the network laws may be applied. The difficulty is that one cannot begin to describe the behaviour of a practical system without setting out, in some depth, the nature of the special elements which may be involved—whether they be machine, or valve or transistor. That, however, is not possible in this brief discourse on circuits. The cases which follow are, in consequence, a selection which requires no special knowledge of the components involved.

7.2 d.c. networks: measurement of displacement

Suppose that we consider a uniform wire of high resistance supplied at each end from a constant voltage source. If a movable contact bears on the wire the voltage between the contact and one end of the wire will be related to the position of the contact (Figure 7.1).

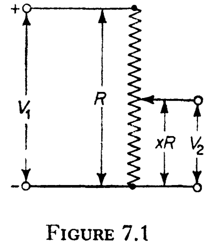


FIGURE 7.1

This can readily be shown, for if

V_1 = input voltage
V_2 = output voltage
I = current in the wire
x = distance of the sliding contact from one end expressed as a fraction of the total length

then, assuming that no current is drawn from the contact terminal

$$V_1 = IR$$
$$V_2 = IxR$$

or

$$\frac{V_2}{V_1} = x$$

so that if the contact is halfway along the wire the output voltage is half the input voltage. This principle is used in slide wire bridge type measurements.

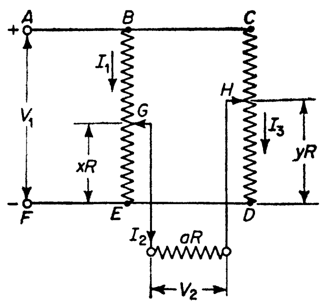


FIGURE 7.2

Now, extending this idea, suppose that there are two identical wires having sliding contacts. In this case, however, a resistor is connected between the contact terminals as shown in the diagram (Figure 7.2). The resistance of each wire is R and that of the third resistor (aR) where a can have any arbitrary value between zero and infinity. Then assuming the current distribution I_1, I_2 and I_3 as indicated, Kirchhoff's first and second laws yield the three equations:

$$\begin{aligned} V_1 &= I_1R(1 - x) + (I_1 - I_2)Rx && \text{Circuit } ABEF \\ &= I_1R - I_2xR \\ O &= I_1R(1 - x) + I_2aR - (I_3 - I_2)R(1 - y) && \text{Circuit } BGHC \\ &= I_1(1 - x) + I_2(a + 1 - y) - I_3(1 - y) \\ V_1 &= (I_3 - I_2)(1 - y)R + I_3yR && \text{Circuit } ACDF \\ &= -I_2(1 - y)R + I_3R. \end{aligned}$$

For chosen values of input voltage and contact positions, and for given values of resistances, these are the three equations needed to determine the three unknown currents. If I_2 is determined in general terms it will be found that

$$I_2 = \frac{V_1}{aR} \cdot \frac{(x - y)}{1 + \dfrac{y(1 - y) + x(1 - x)}{a}}.$$

Now the output voltage

$$V_2 = I_2aR$$
$$= \frac{V_1(x - y)}{1 + \dfrac{y(1 - y) + x(1 - x)}{a}}. \tag{1}$$

If a is now made infinite

$$V_2 = V_1(x - y)$$

or

$$\frac{V_2}{V_1} = (x - y).$$

This indicates that the difference in position of the two contacts is given by the measured output voltage expressed as a fraction of the input. It should be noted that if the output voltage is zero it follows that the sliding contacts are in alignment. Thus we have a device which in effect measures one form of quantity (displacement) and converts it into a signal (in this case, a voltage).

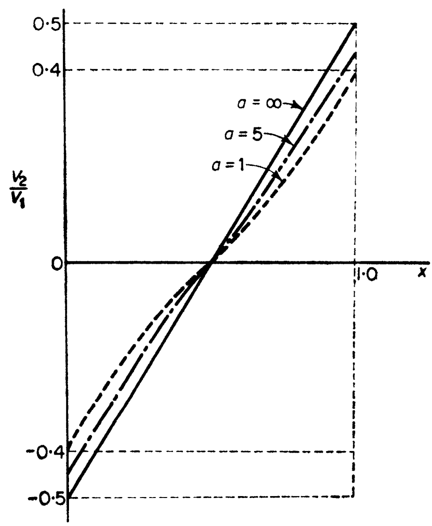


FIGURE 7.3

However, in order to make a measurement of voltage one must have a measuring device between the output terminals. That was why the term aR was included in the circuit; it represents the resistance of the measuring instrument. Provided that a is large—that is, the resistance of the measuring instrument is many times that of the slide wires—there is little error, but if $a \rightarrow 1$—that is, the two resistances are comparable—then the error will be considerable. Some indication of the discrepancy may be obtained by considering the case where $y = 0{\cdot}5$ and varying the value of x. A family of curves of V_2/V_1 against x for various values of a is derived from equation (1). This shows that with diminishing values of a not only is there discrepancy

between the proportions of displacement and voltage but also a departure of the output characteristic from linearity (Figure 7.3).

The resistors have been represented as single uniform conductors of high resistance. One practical form of such a device uses a fine insulated wire wound on a flat strip, or card. The card is then formed into a circle; the sliding contact runs along the edge of the card making contact on a bared surface of wire. Such a 'precision potentiometer' is now able to give an indication of the difference in angular position of two rotatable shafts which carry the contact arms (Figure 7.4).

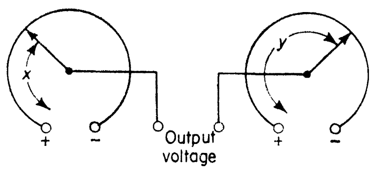


FIGURE 7.4

x and y are now fractions of the total possible angular rotations (something just less than 360°). The pair of potentiometers used in this way might find application in a situation where it is desired to measure, or possibly control, the rotation of a shaft from a remote point. This is the kind of measuring element which would be necessary.

7.3 Network containing non-linear resistors

Suppose that there is a requirement to maintain the voltage of a system at a certain level; the first requirement is a device which can estimate the deviation of this voltage from the desired value. The device must not only be able to assess the magnitude of the deviation but also its sense; that is, whether the voltage is above or below the desired value.

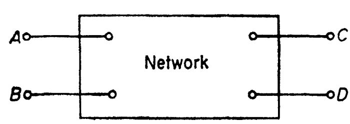


FIGURE 7.5

The requirement is therefore for some kind of network shown diagrammatically above (Figure 7.5). The desired level of voltage, assumed to be from a d.c. source, is applied between A and B. Under these conditions the output voltage between C and D should be zero. If V_{AB} should increase, then V_{CD} should assume a finite value having a certain polarity. If V_{AB} should decrease by a corresponding amount then, ideally, V_{CD} should assume the same finite value as before but be of the opposite polarity. Thus the device is voltage sensitive.

Such a device may be designed in the form of a Wheatstone bridge network. With such a bridge it is common enough practice to have a definite value of input voltage and to adjust the resistance values in

the four arms so that the output voltage is zero; this is known as the condition of balance. But in a conventional bridge, once balance is obtained, altering the value of the input voltage will not upset it. In other words, the conventional Wheatstone bridge is not voltage sensitive.

Now let us take a similar arrangement, but include non-linear resistors in two of the arms of the bridge (Figure 7.6). R_2 and R_2 are non-linear resistors defined by either a mathematical voltage-current relationship or a plotted characteristic which relates the current through the resistor and the voltage across it. R_1 and R_1 are equal, constant, linear resistors.

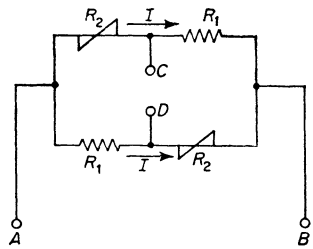


FIGURE 7.6

It is probably best to examine the behaviour of such a device with reference to typical values; the assumption throughout will be that no current is drawn from the output terminals CD.

Let us suppose, therefore, that the required level of direct input voltage is 100 volts, and that the non-linear resistors are defined by the expression

$$V = 30 \times 10^3 \times I^{1\cdot 6}$$

where V is measured in volts and I in amperes. If 100 volts input corresponds to zero volts at the output, it follows that the voltage across each arm of the bridge must be 50 volts. Hence, for either of the non-linear arms (writing I_0 for the current at balance):

$$50 = 30 \times 10^3 \times I_0^{1\cdot 6}$$

or $$I_0 = 18\cdot 3 \times 10^{-3} \text{ amperes.}$$

Under these conditions the current through the linear resistor is exactly the same; it follows that

$$I_0 R_1 = 50$$
$$18\cdot 3 \times 10^{-3} R_1 = 50$$
$$R_1 = 2740 \text{ ohms.}$$

Now in general terms if the current through either branch is I, then the potential of C with respect to D is the potential of C with respect to A plus the potential of A with respect to D

i.e. $$V_{CD} = V_{CA} + V_{AD}.$$

Now $$V_{AD} = IR_1 = 2740I$$

and $$V_{CA} = -30 \times 10^3 I^{1\cdot 6} \qquad \text{(Note the sign)}$$

hence $$V_{CD} = -30 \times 10^3 I^{1\cdot 6} + 2740I.$$

Similarly, considering the input voltage

$$V_{AB} = V_{AC} + V_{CB}$$
$$= +30 \times 10^3 I^{1\cdot6} + 2740I.$$

Also $$\frac{dV_{CD}}{dI} = -30 \times 1\cdot6 \times 10^3 I^{0\cdot6} + 2740$$

and $$\frac{dV_{AB}}{dI} = +30 \times 1\cdot6 \times 10^3 I^{0\cdot6} + 2740$$

that is, $$\frac{dV_{CD}}{dV_{AB}} = \frac{-30 \times 1\cdot6 \times 10^3 I^{0\cdot6} + 2740}{+30 \times 1\cdot6 \times 10^3 I^{0\cdot6} + 2740}.$$

At the condition of balance, when the input voltage is such that there is zero output,

$$I = I_0 = 18\cdot3 \times 10^{-3}A$$

and $$\frac{dV_{CD}}{dV_{AB}} = \frac{-4355 + 2740}{+4355 + 2740} = -0\cdot228.$$

This shows the interesting fact that with this type of bridge the output voltage changes with the input voltage around the condition of balance. In fact, over the whole range of input from zero upwards the characteristic has the form shown below (Figure 7.7). This can be checked by working out input and output voltages for a range of currents above and below the calculated value of I_0.

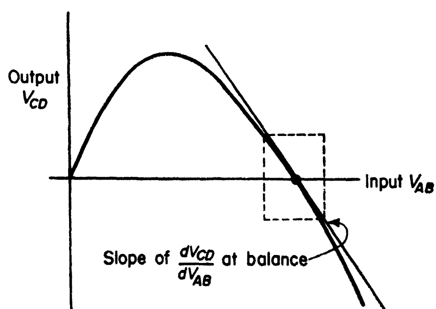


FIGURE 7.7

The example has indicated that such a device has the required form in the neighbourhood of the balance conditions. Here the characteristic has a negative slope of value 0·228 volts per volt. In other words, if the input for some reason falls to 99 volts, we can expect the network to give an output of about +0·23 volts; if it rises to 101 volts, the output from the network will be about −0·23 volts.

Having assessed this deviation it is then necessary to use this information to correct, in some way, the wandering of the input voltage from its required level; but no correction is possible until this assessment of deviation is made—hence the need for such a network.

7.4 Time constant: application of the study of transients

It has already been shown that if a sudden, constant value of voltage is applied to a series circuit comprising a resistor and capacitor the voltage across the capacitor will rise exponentially with time (Figure 7.8). The time of response will depend on the product RC, the time

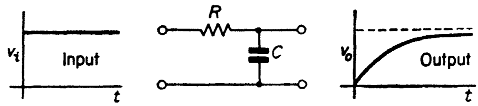


FIGURE 7.8

constant of the circuit. Now suppose that such a circuit is provided with a switch in parallel with the capacitor and each time that the capacitor voltage reaches a predetermined level the switch is closed and immediately opened again (Figure 7.9). The closing of the switch causes instantaneous discharge of the capacitor round the local circuit, reducing the capacitor voltage to zero. The immediate opening of the switch leaves an uncharged capacitor in series with resistance, and build up of capacitor voltage V_c again commences.

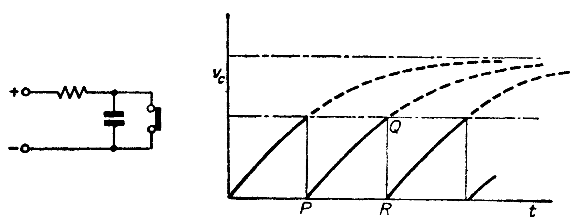


FIGURE 7.9

Provided that the switch closes (and opens) at precisely the same instant in each 'cycle' there will be a saw-tooth form of wave developed.

PQ represents the gradual build up of capacitor voltage.

QR represents the instantaneous discharge.

The time between each repetition will be dependent on the value of voltage at which the capacitor is discharged and the time constant of the circuit. If the discharge is initiated at about, say, one third of the input voltage value, the *PQ* part of the cycle will be reasonably linear.

The capacitor voltage waveform may then be said to increase uniformly with time.

The 'switch' in practice will be a particular form of thermionic valve which is set to 'fire' at the predetermined level of voltage applied to it. It may be taken that when this happens the valve virtually short circuits the capacitor and immediately, because of the reduction of capacitor voltage, reverts to its non conducting state.

This is the basic principle of the time-base of a cathode ray oscilloscope. In such a device a beam of electrons is made to pass between

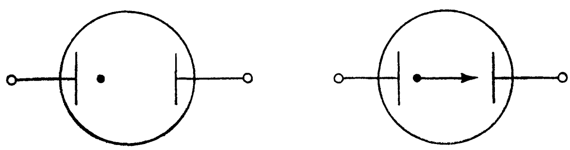


FIGURE 7.10

a pair of vertical conducting plates mounted in an evacuated glass tube. The beam impinges on a prepared surface at the end of the tube. To the observer it appears as a small spot. If a voltage of the form described is applied to the XX terminals the field set up within the tube causes the spot to move sideways at a uniform rate. When the capacitor discharges the spot flies back to its original position. If the frequency of the saw tooth voltage is high enough the first movement of the spot gives the observer the impression that he is looking at a continuous horizontal line. The spot completes its forward travel in, say, milliseconds, and its flyback in microseconds (Figure 7.10).

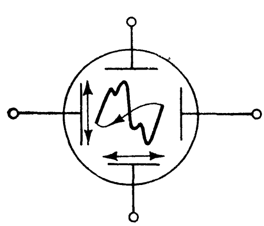


FIGURE 7.11

If some arbitrary voltage of a certain waveform is applied to a pair of horizontal plates in the tube the electron beam is subjected to two movements and if the time base frequency is synchronised with that of the arbitrary voltage, a stationary trace of the waveform appears on the tube (Figure 7.11).

Practical arrangements of a time base circuit are necessarily somewhat complex but all are based on the gradual charge, and instantaneous discharge, of a capacitor.

7.5 Series a.c. circuits

In the following example there is a simple illustration of the application of a.c. circuit theory to a practical case. It concerns the develop-

ment of a circuit to ensure that a particular type of lamp is used in the most economic way.

A commercial type of lamp requires 80 watts for satisfactory working. The voltage which needs to be applied to the lamp in order to produce this condition is 115 volts but the standard supply voltage is 240 volts, 50 cycles per second. The question is, how may the lamp circuit be adapted to suit the standard supply?

At first sight the answer is perfectly straightforward; all that seems to be required is a series resistor in which the voltage drop is the balance of (240 − 115) volts = 125 volts. The lamp current (the lamp is regarded as a pure resistance throughout) is given by

$$I = \frac{P}{V} = \frac{80}{115} = 0{\cdot}7 \text{ amperes}$$

hence the required value of resistance is

$$R = \frac{125}{0{\cdot}7} = 178 \text{ ohms.}$$

However, associated with the series resistance is a power loss of

$$P = (0{\cdot}7)^2 \times 178 = 87{\cdot}2 \text{ watts.}$$

In other words, more power is lost in the resistor than is supplied to the lamp; the total power input would be about 167 watts instead of the 80 watts required. Obviously this is not an economic solution.

However, it has already been pointed out that an inductively wound coil offers a certain reactance to alternating current with the added advantage that there is no dissipation of power in the coil. This is not entirely true because inevitably there will be a certain amount of resistance in the winding; but it is possible to keep this resistance small compared with the inductive reactance. Neglecting this resistance, then, the circuit will take the form of Figure 7.12. The lamp is unaffected in that it requires 80 watts at 115 volts, corresponding to 0·7 amperes, for satisfactory operation. It is now required to estimate the value of inductive reactance required.

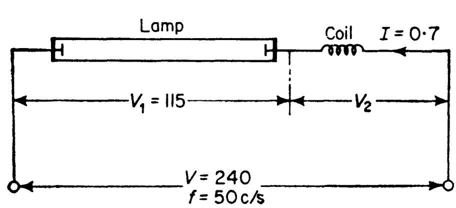


FIGURE 7.12

The circuit yields the following vector diagram (Figure 7.13). Note that the volt drop in inductive reactance leads the current through it by $\pi/2$. The supply voltage is now the vector sum $(\bar{V}_1 + \bar{V}_2)$ and not the arithmetic sum.

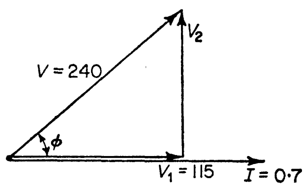


FIGURE 7.13

Solving the voltage triangle

$$V_1^2 + V_2^2 = V^2$$

$$V_2^2 = 240^2 - 115^2$$

$$V_2 = 211 \text{ volts.}$$

Since $IX = V_2$

$$X = \frac{211}{0{\cdot}7} = 301 \text{ ohms}$$

and the inductance of the winding is given by

$$X = \omega L$$

or

$$301 = 2\pi \times 50 \times L$$

$$L = 0{\cdot}96 \text{ henrys.}$$

Now suppose that a modification is made to the circuit by connecting a capacitor in parallel with the series arrangement just discussed (Figure 7.14).

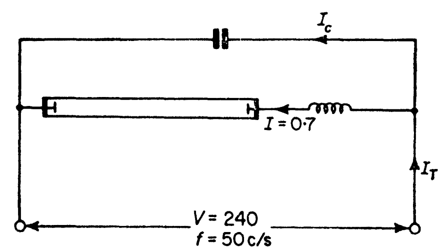


FIGURE 7.14

7.6 Parallel circuit

The previous vector diagram, for the lamp circuit, is unaffected by the addition of the capacitor; but account must be taken of the current flowing in the capacitor. This will lead the voltage across the capacitor by $\pi/2$. Adding this to the previous diagram will give a complete vector diagram of the form shown in Figure 7.15.

Note that φ, in this diagram is given by

$$\tan^{-1}(211/115) = 61{\cdot}5°.$$

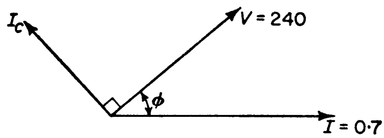


FIGURE 7.15

Leaving out the V_1 and V_2 vectors, which were relevant only to the last part of the question, the total input current may be determined from the vector summation of the two components, that is $\bar{I}_T = \bar{I} + \bar{I}_c$ (shown in Figure 7.16).

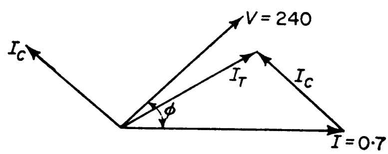


FIGURE 7.16

For if example

$$\begin{aligned} C &= 8 \times 10^{-6} \text{ farads} \\ I_c &= V\omega C \\ &= 240 \times 2\pi \times 50 \times 8 \times 10^{-6} \text{ amperes} \\ &= 0{\cdot}6 \text{ amperes} \end{aligned}$$

and solution of the current triangle shows that

$$I_T = 0{\cdot}34 \text{ amperes.}$$

That is, the input current to the whole circuit is less than that taken by the lamp circuit by itself. The current carrying capacity of the supply to the lamp can therefore be less if such a capacitor is connected. This may not seem significant in the case of a single lamp but in an installation where a large number of such lamps are connected it is an important consideration.

The example taken relates to a common type of fluorescent lighting unit where one is provided not only with a lamp but with a 'control' circuit. This circuit includes the series inductor and the shunt capacitor. The values taken are not quite representative since two assumptions have been made. One is that winding resistance has been neglected in estimating the value of the coil inductance, the other is that such a lamp can be represented as a linear resistor.

7.7 Application of the second order differential equation: control systems

The subject of control is a study in its own right and one which has received increasing attention in recent years. One does not have to look far to see the reasons. In all branches of engineering the accurate control of temperature, voltage or current, or the speed or displacement of some part of a system, has assumed tremendous importance. It is not inappropriate to examine briefly some of the basic considerations necessary to such a study. Electrical networks and control systems have one very important factor in common; at the heart of any problem in each of these areas of study is the need to formulate the differential equation which describes the behaviour of the system.

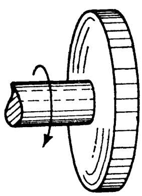


FIGURE 7.17

There is another, more obvious, overlap between the two studies; many control systems are either entirely, or in part, made up of electrical components. Already note has been made of two networks, one of which can measure displacement whilst the other senses a voltage variation. Such networks could be the basic measuring elements in the appropriate control sequence.

Suppose that there is a requirement for the accurate positioning of a shaft to which is coupled a member having considerable inertia and further suppose that the torque necessary to turn the shaft at a constant rate is directly proportional to that rate (Figure 7.17). Then let:

J = the system inertia

θ_0 = the shaft rotation from some datum position at any instant

$\mathrm{d}\theta_0/\mathrm{d}t$ = the shaft speed at that instant and

$\mathrm{d}^2\theta_0/\mathrm{d}t^2$ = the shaft acceleration at that instant.

The torque acting on the shaft at this instant of time will be made up of two components: a component to produce the angular acceleration and a component to take account of the torque necessary to run the shaft at the given speed. Thus

$$T = J\frac{\mathrm{d}^2\theta_0}{\mathrm{d}t^2} + K\frac{\mathrm{d}\theta_0}{\mathrm{d}t}.$$

Now suppose that this shaft is to follow faithfully the angular position of another shaft situated in a remote place. (One could imagine this as being the requirement on a ship, where the accurate positioning

of a massive rudder is to be controlled by the movement of something akin to a compass pointer on the bridge.) The next requirements are:

(i) Some means of comparing the angular positions of the two members, and

(ii) A means of converting the angular difference between them into an appropriate signal.

An earlier example has shown that a pair of precision potentiometers will perform both of these functions. One would be attached to the 'master' position whilst the other would be coupled to the controlled member. The signal in this case would be a direct voltage.

There would then be a requirement to convert this signal to a torque applied to the controlled shaft. This might take the form of an amplifier the output of which supplies a motor geared to the shaft. Such a system is said to incorporate 'feedback' because information at the output end of the system is continuously fed back and compared with the input.

It will be assumed that the torque developed at the controlled shaft is directly proportional to the angular difference which exists. That is

$$T \propto (\theta_i - \theta_0)$$

where θ_i is the input angular displacement or

$$T = G(\theta_i - \theta_0).$$

The complete equation may then be formed

$$G(\theta_i - \theta_0) = J\frac{\mathrm{d}^2\theta_0}{\mathrm{d}t^2} + K\frac{\mathrm{d}\theta_0}{\mathrm{d}t}.$$

J and K will now not merely relate to the output member but include the effects of the driving motor and its associated gear train referred, for convenience, to the output member. The equation reduces to

$$J\frac{\mathrm{d}^2\theta_0}{\mathrm{d}t^2} + K\frac{\mathrm{d}\theta_0}{\mathrm{d}t} + G\theta_0 = G\theta_i$$

which may be written in terms of the parameters ω_n and ζ

$$\frac{\mathrm{d}_2\theta_0}{\mathrm{d}t^2} + 2\zeta\omega_n\frac{\mathrm{d}\theta_0}{\mathrm{d}t} + \omega_n^2\theta_0 = \omega_n^2\theta_i$$

where $\omega_n^2 = G/J$ and $2\zeta\omega_n = K/J$. This differential equation describes precisely the behaviour of the system. If the system input has a datum position of $\theta_i = 0$ but the output member is, for some reason, held at some value different from this and then released, the

subsequent output movement is governed by the values of ω_n and ζ which are, in turn, related to the system constants (Figure 7.18).

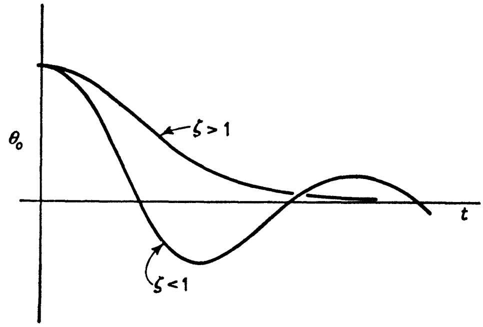


FIGURE 7.18

This same characteristic behaviour has already been discussed in connexion with a mechanical system and its analogous network. It points to the fact that in such a feedback system the 'damping' term represented by the constant K is most significant. If it were absent, then any disturbance would result in simple harmonic motion of the output member.*

Necessarily, a practical position control system is somewhat more complex than that envisaged here. This is partly because of the fact that each element which makes up a sequence may introduce additional terms, probably time-dependent, into the differential equation. Another reason is that the form of damping inherent in the output member as indicated in the above example may be unacceptable. In practise the amplifier part of the sequence may include networks which by virtue of their characteristics introduce particular forms of damping term into the differential equation. Nevertheless the possibility of an oscillatory form of response as outlined above is characteristic of such feedback systems.

* The solution of such a differential equation is included in Appendix I.

Appendix I

SOLUTION OF A SECOND ORDER DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

The solution of an equation of the form

$$\tau \frac{dx}{dt} + x = 0$$

may be determined by a process of separating the variables; the problem then reduces to a simple integration

$$\tau \frac{dx}{dt} = -x.$$

Integrating $$\int \frac{dx}{x} = \int -\frac{dt}{\tau} + \log_e A$$

where $\log_e A$ is the necessary constant of integration. It follows that

$$\log_e x = -\frac{t}{\tau} + \log_e A$$

$$\log_e \frac{x}{A} = -\frac{t}{\tau}$$

or $$x = Ae^{-t/\tau}. \tag{1}$$

The constant of integration can only be determined from a knowledge of the value of x at any particular instant in time. If for example $x = X_0$ at a time $t = 0$ then substituting in (1)

$$X_0 = Ae^0$$

or $$A = X_0$$

and the final solution is $x = X_0 e^{-t/\tau}$.

This solution might have been reached by other methods. A study of the equation

$$\tau \frac{dx}{dt} + x = 0$$

shows that the solution must take the form of a quantity which when differentiated once has exactly the same form.

$$x = e^{at}$$

is one such function since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ae^{at}.$$

Intuitive thinking on these lines would show that

$$x = Ae^{-t/\tau}$$

is the required function since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{A}{\tau}e^{-t/\tau}$$

and substitution in the original differential equation gives

$$\tau\left(-\frac{A}{\tau}e^{-t/\tau}\right) + Ae^{-t/\tau} = 0.$$

Thus the equation is satisfied by this form of function. Determination of the constant of integration follows exactly as before.

Now suppose that a second order differential equation with constant coefficients is considered.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_n\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_n^2x = 0.$$

Just as one constant of integration was necessary in the first case, since effectively, the solution of the equation involved one integration, so in this case there are two integrations implicit in the solution and two constants of integration are necessary. Again, the form of the solution must be such that successive differentiation results in a function of the same form.

Suppose that the solution is

$$x = Ae^{k_1t} + Be^{k_2t}. \qquad (2)$$

Then
$$\frac{\mathrm{d}x}{\mathrm{d}t} = k_1Ae^{k_1t} + k_2Be^{k_2t}$$

and
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = k_1^2Ae^{k_1t} + k_2^2Be^{k_2t}.$$

Substitution in the differential equation gives

$$Ae^{k_1t}(k_1^2 + 2\zeta\omega_nk_1 + \omega_n^2) + Be^{k_2t}(k_2^2 + 2\zeta\omega_nk_2 + \omega_n^2) = 0.$$

This can only be so if

$$k_1^2 + 2\zeta\omega_nk_1 + \omega_n^2 = 0$$

and
$$k_2^2 + 2\zeta\omega_nk_2 + \omega_n^2 = 0.$$

That is if
$$k_1 = -\zeta\omega_n + \omega_n\sqrt{(\zeta^2 - 1)}$$

and
$$k_2 = -\zeta\omega_n - \omega_n\sqrt{(\zeta^2 - 1)}.$$

It should be noted that k_1 and k_2 cannot be the same, as this would effectively reduce to the fact that there is only one constant of integration in the solution; hence the opposite signs in the second term.

Carrying all these terms through would make for a rather unwieldy form of solution; k_1 and k_2 will therefore be used in subsequent work instead of their values in terms of ζ and ω_n. Again, it is necessary to take account of initial conditions before a final solution may be found. Suppose these are that when $t = 0$, $x = X_0 =$ constant, and $\mathrm{d}x/\mathrm{d}t = 0$.

This form of equation was encountered when dealing with the torsional pendulum. The conditions set out above simply imply that the system is given an initial angle of twist (corresponding to $x = X_0$) and that it has zero velocity ($\mathrm{d}x/\mathrm{d}t = 0$) before being freed at $t = 0$.

The first condition is that at $t = 0$, $x = X_0$. Substituting this in Equation (2)

$$\begin{aligned} x &= Ae^{k_1 t} + Be^{k_2 t} \\ X_0 &= Ae^0 + Be^0 \\ X_0 &= A + B. \end{aligned} \tag{3}$$

To apply the second condition it is necessary to differentiate Equation (2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k_1 Ae^{k_1 t} + k_2 Be^{k_2 t}$$

at $t = 0$

$$0 = k_1 A + k_2 B. \tag{4}$$

From Equations (3) and (4) it follows that

$$A = -\frac{k_2}{k_1 - k_2} X_0$$

and

$$B = +\frac{k_1}{k_1 - k_2} X_0.$$

The final solution, substituting these constants in Equation (2) is

$$x = \frac{X_0}{(k_1 - k_2)} (k_1 e^{k_2 t} - k_2 e^{k_1 t}). \tag{5}$$

Substitution for k_1 and k_2 will give the value of x as a function of t in terms of the parameters ζ and ω_n.

A particularly interesting form of equation arises if ζ is less than unity. In this event $\omega_n\sqrt{(\zeta^2 - 1)}$ may be written as $\omega_n\sqrt{[(-1)(1 - \zeta^2)]}$ or as $j\omega_n\sqrt{(1 - \zeta^2)}$.

The values for k_1 and k_2 now become

$$k_1 = -\zeta\omega_n + j\omega_n\sqrt{(1-\zeta^2)} = -a + jb$$
$$k_2 = -\zeta\omega_n - j\omega_n\sqrt{(1-\zeta^2)} = -a - jb$$

where

$$a = \zeta\omega_n$$
$$b = \omega_n\sqrt{(1-\zeta^2)}.$$

Substitution in the general form of the equation for x (Equation (5)) gives

$$x = \frac{X_0}{2jb}\left[(-a+jb)e^{(-a-jb)t} - (-a-jb)e^{(-a+jb)t}\right]$$
$$= \frac{X_0}{2jb}e^{-at}\left[-a(e^{-jbt} - e^{+jbt}) + jb(e^{-jbt} + e^{+jbt})\right].$$

It can be shown that

$$\frac{(e^{jbt} - e^{-jbt})}{2j} = \sin bt \quad \text{and} \quad \frac{(e^{jbt} + e^{-jbt})}{2} = \cos bt$$

so that the function x may be written

$$x = \frac{X_0}{b}e^{-at}(a \sin bt + b \cos bt).$$

This may be written as

$$x = (a^2 + b^2)^{1/2}\frac{X_0}{b}e^{-at}\left[\frac{a}{(a^2+b^2)^{1/2}}\sin bt + \frac{b}{(a^2+b^2)^{1/2}}\cos bt\right].$$

By doing this the bracketed term is a simple expansion, if an angle is defined as $\alpha = \tan^{-1}b/a$. For then,

$$\sin\alpha = \frac{b}{(a^2+b^2)^{1/2}}$$

and

$$\cos\alpha = \frac{a}{(a^2+b^2)^{1/2}}$$

and the equation becomes

$$x = \frac{X_0}{b}e^{-at}(a^2+b^2)^{1/2}(\sin bt \cos\alpha + \cos bt \sin\alpha)$$
$$= \frac{X_0}{b}e^{-at}(a^2+b^2)^{1/2}\sin(bt+\alpha).$$

In this equation

$$(a^2+b^2)^{1/2} = \omega_n$$
$$b = \omega_n\sqrt{(1-\zeta^2)}$$
$$a = \zeta\omega_n$$
$$\alpha = \tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

and hence in terms of the original parameters

$$x = \frac{X_0}{\sqrt{(1 - \zeta^2)}} e^{-\zeta\omega_n t} \sin \left[\omega_n \sqrt{(1 - \zeta^2)}t + \tan^{-1} \frac{\sqrt{(1 - \zeta^2)}}{\zeta}\right]. \quad (6)$$

If the particular value of $\zeta = 0$ is taken, then this represents a system without damping. Substituting in the above equation results in the response

$$x = \frac{X_0}{1} e^0 \sin (\omega_n t + \tan^{-1} \infty)$$

$$= X_0 \sin \left(\omega_n t + \frac{\pi}{2}\right)$$

or

$$= X_0 \cos \omega_n t.$$

That is, the system would oscillate at its natural undamped frequency.

Equation (6) is that of a function in which the angular frequency $\omega_n \sqrt{(1 - \zeta^2)}$ is constant but the amplitude decays exponentially with time (Figure 3.4). Note that the solution indicates that at $t = 0$, $x = X_0$, the specified initial condition.

The case illustrated is for a low value of ζ. As ζ is increased the oscillation becomes more heavily damped until at $\zeta = 1$ there is no trace of oscillation whatsoever. It so happens that this 'critical' case cannot be derived from the general solution. (Try putting $\zeta = 1$ in Equation (6).) This is a case where the roots k_1 and k_2 are real and equal and the solution

$$x = Ae^{k_1 t} + Be^{k_2 t}$$

gives (if $k_1 = k_2 = k$)

$$x = (A + B)e^{kt},$$

which is only effectively including one constant of integration, whereas with the second order equation two are necessary.

It can be shown that for this particular case the only solution which obtains is

$$x = (A + Bt)e^{kt}$$

where $k = -\omega_n$. The constants A and B may be determined from initial conditions in exactly the same way as before.

Appendix II

A NOTE ON THE MATRIX NOTATION

Passing reference was made in dealing with simultaneous equations to matrix methods. This field of mathematics is a special subject of study and a note such as this cannot do more than give the briefest of introductions to the matrix notation.

A matrix is an array of numbers in rows and columns. An example of a 2 × 3 (two by three) matrix is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}.$$

That is, the matrix comprises two rows and three columns. In more general terms an *m* by *n* matrix is an array of *mn* terms comprising *m* rows and *n* columns. Such notation is useful in handling and solving sets of equations. Suppose for example that we consider two equations relating the variables u and v with the variables x and y.

$$u = ax + by$$
$$v = cx + dy.$$

Then the conventional matrix representation of these equations would be

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The relationship between this form and the actual equations is probably more clearly seen if arrows are inserted thus

$$\begin{bmatrix} u \\ v \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \overset{\longrightarrow}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} \Big\downarrow$$

The equations represented by the matrix equation can be obtained by taking the first row of the centre 'box', multiplying each of its elements by the corresponding element in the right hand column, and adding. The same process is then applied to the second row. A detailed knowledge of the terminology and manipulations associated with this notation is only possible after a special study of the subject; but the following example may give some indication of the application of

the notation to circuit theory. The example relates to the method of finding the product of two matrices.

Suppose that the two equations

$$\begin{aligned} u &= a_1x + b_1y \\ v &= c_1x + d_1y \end{aligned} \qquad (1)$$

relate u and v to two other variables x' and y' through another pair of equations

$$\begin{aligned} x &= a_2x' + b_2y' \\ y &= c_2x' + d_2y'. \end{aligned} \qquad (2)$$

Then by substituting for x and y from (2) into (1) it can be seen that

$$\begin{aligned} u &= (a_1a_2 + b_1c_2)x' + (a_1b_2 + b_1d_2)y' \\ v &= (c_1a_2 + d_1c_2)x' + (c_1b_2 + d_1d_2)y' \end{aligned} \qquad (3)$$

thus showing a direct relationship between u and v and x' and y'.

Now suppose that Equations (1) and (2) had been set out in matrix form.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Hence

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Now Equation (3) shows that in matrix form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

which indicates that

$$\overrightarrow{\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\Big\downarrow = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}.$$

Again, the direction of arrows gives a lead to the sequence of formation of products on the right hand side of the matrix equation.

There may not appear to be any great point in making this kind of transformation. Basically what has happened is that the logical mathematical development from Equations (1) and (2) to (3) has been replaced by a 'machine-like' technique. However we have already seen that networks can give rise to numbers of interdependent equations. Matrix notation assists in the representation, reduction and solution of such equations by providing, through transformations of the type noted above, a routine sequence of operations. It is quite

common practice to describe a particular type of network by relating input and output quantities with the network constants. To take a very simple case (Figure A):

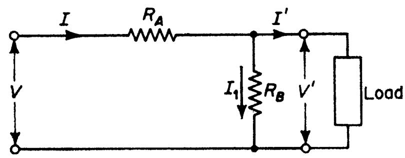


FIGURE A

V and I are the input voltage and current
V' and I' are the output voltage and current into an arbitrary load.

Then
$$I_1 = \frac{V'}{R_B}$$
$$I = I' + \frac{V'}{R_B}$$
and
$$V = V' + IR_A = V' + \left(I' + \frac{V'}{R_B}\right)R_A.$$

That is, the relationship between input and output quantities is given by the pair of equations

$$V = \left(1 + \frac{R_A}{R_B}\right)V' + R_A I'$$
and
$$I = \frac{1}{R_B}V' + I'.$$

These may be written as

$$V = aV' + bI'$$
$$I = cV' + dI'$$

where $a = (1 + R_A/R_B)$, $b = R_A$, $c = 1/R_B$, $d = 1$.

Now suppose that a pair of networks are defined in this fashion (Figure B). Then, setting out these relationships in matrix form

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} V' \\ I' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V' \\ I' \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} V'' \\ I'' \end{bmatrix}.$$

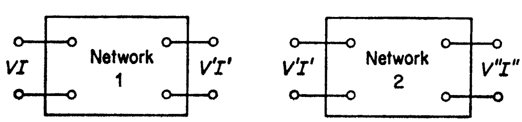


FIGURE B

Now suppose that the output of the left-hand network is joined to the input of the right-hand network. In all probability the voltage and current values will differ from whose which held when the networks were regarded as quite separate and supplying their own loads. However, provided that the principle of superposition holds, the actual constants are the same as before and the complete network shown in Figure C results in the matrix equation

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \begin{bmatrix} V'' \\ I'' \end{bmatrix}.$$

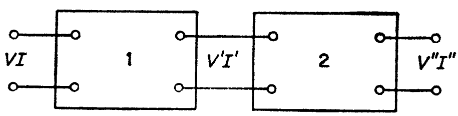


FIGURE C

Thus a complete picture of the constants resulting from 'cascading' two networks may be readily obtained. Such a manipulation is not of course confined to resistance networks of the type used to illustrate this example. The coefficients may very well relate to impedance values expressed in complex form when a.c. circuits are considered.

ANSWERS TO EXERCISES

Chapter 1

1 (a) 100 volts (b) 12·5 amperes (c) 22·5 amperes
(d) 212·5 volts

2 (a) 0·5 ohms (b) $1{\cdot}33 \times 10^{-3}$ amperes (c) 0·06 volts
(d) 0·966 volts

3 (a) 13/8 ohms (b) 80/13 amperes

4 (a) 0·5 amperes (b) $-0{\cdot}5$, $-0{\cdot}5$, and 1·5 amperes
(c) 1·25 watts

5 (a) $12{\cdot}5 \times 10^{-3}$ amperes (b) $10{\cdot}7 \times 10^{-3}$ amperes
(c) $8{\cdot}3 \times 10^{-3}$ amperes

6 Points on the characteristic

Total current (milliamperes)	6·5	16·0	28·5	44·0
Total voltage	5·95	15·8	29·55	47·2

Chapter 2

1 (a) $i = 0{\cdot}05(1 - e^{-200t})$ amperes
(b) $v_R = 10(1 - e^{-200t})$ volts, $-L(\mathrm{d}i/\mathrm{d}t) = -10e^{-200t}$ volts
(c) 0·0316 amperes
(d) 10 amperes per second, 3·68 amperes per second
(e) $4{\cdot}22 \times 10^{-4}$ joules

2 (a) $q = 1 \times 10^{-6}(1 - e^{-5t})$ coulombs
(b) $0{\cdot}632 \times 10^{-6}$ coulombs (c) $i = 5 \times 10^{-6}e^{-5t}$ amperes
(d) 5×10^{-6} amperes, $1{\cdot}84 \times 10^{-6}$ amperes
(e) $v_R = 10e^{-5t}$ volts (f) $v_C = 10(1 - e^{-5t})$ volts
(g) $4{\cdot}33 \times 10^{-6}$ joules

3 (a) $q = 1 \times 10^{-6}(1 - 0{\cdot}2e^{-5t})$ coulombs
(b) $i = 1 \times 10^{-6}e^{-5t}$ amperes

Chapter 3

1 (a) $q = 20 \times 10^{-6} \cos 5 \times 10^3 t$ coulombs
(b) $v_C = 5 \cos 5 \times 10^3 t$ volts

(c) $i = -0{\cdot}1 \sin 5 \times 10^3 t$ amperes
(d) $-L(\mathrm{d}i/\mathrm{d}t) = 5 \cos 5 \times 10^3 t$ volts
(e) $10^4/4\pi$ cycles per second
(f) 50×10^{-6} joules, $14{\cdot}6 \times 10^{-6}$ joules

2 (a) 0·5 (b) 689 cycles per second (c) 50 ohms

Chapter 4

1 (a) $v_1 = 10 \sin 2000\pi t$ volts, $v_2 = 8 \sin (2000\pi t - \pi/4)$ volts
(b) $\pi/4$ radians

2 (a) $v = 4{\cdot}0 \sin 314t$ volts (b) $v = 12{\cdot}56 \sin (314t + \pi/2)$ volts
(c) $v = 159 \sin (314t - \pi/2)$ volts

3 (a) 5 volts (b) 2·5 volts (c) 2·88 volts

4 (a) $p = 10 \cos \pi/3 - 10 \cos (4000\pi t - \pi/3)$ watts
(b) 5 watts (c) 0·5

Chapter 5

1 (a) $22{\cdot}6 \sin (\omega t + 0{\cdot}392)$ volts (b) 22·6 volts
(c) 16·0 volts

2 (a) 628 ohms (b) 696 ohms (c) 0·0145 amperes
(d) 1·125 radians (e) $0{\cdot}179 \times 10^{-3}$ seconds
(f) 4·32 volts (g) 9·04 volts (h) 0·0622 watts

3 (a) $2{\cdot}53 . 10^{-6}$ farads (b) 1·0 ampere (c) 10 volts
(d) 62·8 volts (e) 0·105 amperes (f) 1·05 volts
(g) 3·3 volts (inductance), 13·2 volts (capacitance)
(h) 1·47 radians

4 (a) 1·2 amperes (b) 1·87 amperes (c) 2·22 amperes
(d) 0·99 radians (e) 24 watts

5 (a) $15{\cdot}9 \times 10^{-6}$ farads (b) 0·198 amperes
(c) 50·6 ohms

Chapter 6

1 $(1/\sqrt{2})(20 + j0)$, $(1/\sqrt{2})(1{\cdot}732 - j1{\cdot}0)$, $(8{\cdot}66 + j5{\cdot}0)$ ohms

2 (a) 32·02 volts (b) 6·4 amperes (c) 10·3 amperes
(d) 51·5 volts (e) 67·1 volts
(f) 100 watts, 205 watts, 318 watts (g) 623 watts
(h) $5{\cdot}88 - j2{\cdot}82$ ohms

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